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Semisimplicity and Reduction of p-adic Representations of Topological Monoids

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Abstract

We study relations between several p-adic variants of the semisimplicity of Banach algebras, Banach modules, unitary Banach representations, and the reductions of them. We give a criterion of the semisimplicity of a p-adic unitary representation of a topological monoid by using the reduction of the associated operator algebra. It yields an algorithm for determining whether a given finite dimensional p-adic unitary Banach representation of a compact p-adic Lie group is presentable as the orthogonal direct sum of absolutely irreducible subrepresentations or not.

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Introduction

Let k be a complete valuation field with valuation ring $k(1)$ and residue field $\overline{k}$, and G a topological monoid. Unlike a unitary $\mathbb{C}$-linear representation of a locally compact group, a Banach k-linear representation of G is not necessarily completely reducible even though suitable conditions on G and the representation are assumed. This causes difficulty in an appropriate formulation of a C^*-algebra and a von-Neumann algebra over k. We are interested in operator algebra theory in the non-Archimedean setting, and seek a formulation of such operator algebras associated to G possessing much information on semisimplicity representations. We note that there are several formulation of operator algebras associated to G under suitable conditions such as the Iwasawa algebra associated to a profinite group (cf. [ST02] Theorem 2.3) and the three types of multiplier Banach–Hopf algebras over k associated to several topological groups (cf. Introduction in [Koc]), and we seek another formulation which specialises in semisimple representations.

For this purpose, we study the semisimplicity of a finite dimensional Banach k-linear representation using tools which are only applicable to the non-Archimedean setting, e.g. the reduction. It is difficult to determine the semisimplicity of a unitary Banach k-linear representation of G from the information of its reduction, because it only reflects the action of the reduction $\overline{k}[G]$ of the canonical integral model $k^\circ[G] \subset k[G]$. On the other hand, for a given unitary Banach k-linear representation V of G, the operator algebra $\mathrm{C}^*(G, V)$ given as the closure of the image of $k[G]$ in the full operator algebra $\mathscr{B}(V)$ admits a canonical integral model for which the reduction possesses much more information on the semisimplicity.

The aim of this paper is to give an algorithm (RR) for computing the reduction of $\mathrm{C}^*(G, V)$ and determining whether a given finite dimensional unitary Banach k-linear representation of G is semisimple in a strong sense, i.e. is presentable as the orthogonal direct sum of absolutely irreducible subrepresentations under the assumption that k is a local field and G lies in a certain class of profinite groups containing that of compact p-adic Lie groups. See Theorem 3.16. We also have an algorithm (R1) for determining whether a given finite dimensional unitary Banach k-linear representation of G is absolutely irreducible under the same assumption. See Theorem 3.14. We note that the algorithm (RR) for the semisimplicity in the strong sense is a generalisation of the algorithm for the unitary diagonalisability of a matrix in [Mih16] Theorem 2.23. Indeed, an $M \in \mathrm{M}_n(k)$ with $n \in \mathbb{N} \setminus \{0\}$ is diagonalisable by a unitary matrix (cf. [Mih16] p. 762) if and only if the n-dimensional unitary Banach k-linear representation $\mathbb{Z}_p \times k^n \to k^n$, $(i, v) \mapsto (1 + \pi_k^r M)^i v$ of the compact p-adic Lie group $\mathbb{Z}_p$ is presentable as the orthogonal direct sum of absolutely irreducible subrepresentations for a sufficiently large $r \in \mathbb{N} \setminus \{0\}$, where π_k is a

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uniformiser of the local field k.

In order to give an algorithm, we study relations between several variants of the semisimplicity of Banach k-algebras, Banach modules over them, unitary Banach k-linear representations, and the reductions of them. Compared with the variants of the semisimplicity of unitary Banach k-linear representations, those of Banach k-algebras and Banach modules over them behave very well with respect to the reductions. For example, the theory on the lifting property of idempotents with respect to the quotient modulo a regular ideal in [Azu51] Theorem 24 helps us to decompose Banach k-algebras, and decompositions of Banach k-algebras yield decompositions of Banach modules over them.

We note that the semisimplicity of a finite dimensional Banach k-linear representation is itself interesting in the context away form the study on operator algebra theory, and is helpful for an explicit computation because the decomposition into the direct sum of isotypic components reduces the dimension. For example, there are several well-known variants of conjectures called the semisimplicity conjecture (cf. Introduction in [Fu01]), which state the semisimplicity of representations related to Galois representations, e.g. actions of Frobenius automorphisms.

We explain the contents of this paper. First, §1 consists of three subsections. In §1.1, we recall theory on semisimple modules. In §1.2, we study semisimple modules whose simple submodules are absolutely simple, and recall semisimple smooth representations. In §1.3, we recall the semisimplicity and the reductions of unitary Banach k-linear representations. Next, §2 consists of two subsections. In §2.1, we recall Banach k-algebras and Banach modules over them, and introduce several variants of the semisimplicity of them. In §2.2, we introduce operator algebras associated to Banach modules over Banach k-algebras, and study relations between the semisimplicity of Banach modules and operator algebras associated to them. Finally, §3 consists of three subsections. In §3.1, we study the relation of the semisimplicity of Banach k-algebras and of their reductions. In §3.2, we study the relation of the semisimplicity of Banach modules and of their reductions regarded as modules over the reduction of the operator algebras associated to them. In §3.3, we give an algorithm for determining whether a given finite dimensional unitary Banach k-linear representation is presentable as the orthogonal direct sum of absolutely irreducible subrepresentations.

1 Preliminaries

We recall semisimple modules and p-adic unitary Banach representations. For this purpose, we prepare the conventions. Let Set denote the category of sets and maps, and Ring the category of rings and ring homomorphisms. Here we assume that a ring is unital and associative, but is not necessarily commutative. For an $F \in \mathrm{ob}(\mathrm{Ring})$, we denote by F^{op} the opposite ring of F, by $\mathrm{Idem}(F) \subset F$ the subset of idempotents, and by $\mathrm{Z}(F) \subset F$ the commutative subring of central elements.

1.1 Simplicity and Semisimplicity

We introduce several variants of the simplicity and the semisimplicity of rings and left modules. Let $F \in \mathrm{ob}(\mathrm{Ring})$. We denote by $\mathrm{Vect}(F)$ the category of left F-modules and F-linear homomorphisms. We abbreviate $\mathrm{Hom}_{\mathrm{Vect}(F)}$ (resp. $\mathrm{End}_{\mathrm{Vect}(F)}$) to Hom_F (resp. End_F). We regard a right F-module V as a left F^{op}-module in a natural way.

Let $V \in \mathrm{ob}(\mathrm{Vect}(F))$. We say that V is *simple* if V admits exactly two left F-submodules, is *isotypic* if V is isomorphic in $\mathrm{Vect}(F)$ to the direct sum of copies of a simple left F-module, is *semisimple of finite type* if V is isomorphic in $\mathrm{Vect}(F)$ to the direct sum of a finite family of isotypic left F-modules, and is *semisimple of finite type* (resp. *semisimple*) if V is isomorphic in $\mathrm{Vect}(F)$ to the direct sum of a finite family (resp. family) of simple left F-modules. We say that F is *simple* if F admits exactly two two-sided ideals, and is *semisimple* if F is a semisimple left F-module.

We put $F'_V := \mathrm{End}_F(V)^{\mathrm{op}} \in \mathrm{ob}(\mathrm{Ring})$ and $F''_V := \mathrm{End}_{\mathrm{End}_F(V)}(V) \in \mathrm{ob}(\mathrm{Ring})$. The scalar multiplication $F \times V \to V$ induces an injective ring homomorphism $\Psi_{V,F} \colon F/\mathrm{Ann}_F(V) \hookrightarrow F''_V$. By $F''_V \subset (F''_V)''_V = \mathrm{End}_{\mathrm{End}_F(V)''_V}(V)$ and $\mathrm{End}_F(V) \subset \mathrm{End}_F(V)''_V$, we have $F''_V = (F''_V)''_V$. The *weak topology* on F''_V is the topology generated by the set $\{\{f' \in F''_V \mid (f' - f)v = 0\} \mid (f, v) \in F''_V \times V\}$. Since $\{f \in F''_V \mid (fv)_{v \in S} = (0)_{v \in S}\}$ forms an open neighbourhood of $0 \in F''_V$ with respect to the weak topology for any finite subset $S \subset V$, we obtain the following:

Proposition 1.1. *If V is finitely generated as a right F'_V-module, then the weak topology on F''_V coincides with the discrete topology.*

By Proposition 1.1 and Jacobson–Bourbaki density theorem (cf. [Cri04] D 2.2), we obtain the following:

Corollary 1.2. *If V is a semisimple left F-module finitely generated as a right F'_V-module, then $\Psi_{V,F}$ is an isomorphism in* Ring.

Suppose that F is commutative. We denote by $\mathrm{Alg}(F)$ the category of F-algebras and F-algebra homomorphisms. Let $A \in \mathrm{ob}(\mathrm{Alg}(F))$ and $V \in \mathrm{ob}(\mathrm{Vect}(A))$. We say that A (resp. V) is *finite dimensional* if A (resp. V) is finitely generated as an F-module. We recall the structure of a finite dimensional simple left A-module.

Proposition 1.3. *Suppose that F is a field and V is finite dimensional. Then the following are equivalent.*

(i) The left A-module V is simple.

(ii) There is a surjective F-algebra homomorphism $\pi \colon A \twoheadrightarrow \mathrm{M}_n(A'_V)$ with $n \in \mathbb{N} \setminus \{0\}$ such that V is isomorphic in $\mathrm{Vect}(A)$ *to $A/\ker(\pi) \otimes_{\mathrm{M}_n(A'_V)} (A'_V)^n$, and A'_V is a non-zero division F-algebra.*

Proof. The implication from (ii) to (i) is obvious. We show the implication from (i) to (ii). Suppose that V is simple. Put $D := A'_V$. Since D is an F-algebra, V is finitely generated as a right D-module, and hence we have an isomorphism $\Psi_{V,A} \colon A/\mathrm{Ann}_A(V) \to A''_V$ in $\mathrm{Alg}(F)$ by Corollary 1.2. By Schur's lemma (cf. [Beh72] II 1 Theorem 2), D is a division F-algebra with $0 < d := \dim_F D \leq (\dim_F V)^2 < \infty$, and a right D-linear basis of V gives an isomorphism $A''_V \to \mathrm{M}_n(D)$ with $n := \dim_D V \in \mathbb{N} \setminus \{0\}$ in $\mathrm{Alg}(F)$. The fixed D-linear basis of V gives an isomorphism $A/\mathrm{Ann}_A(V) \otimes_{\mathrm{M}_n(D)} D^n \to V$ in $\mathrm{Vect}(A)$. □

We recall the relation between the isotypic property of V and the simplicity of the operator algebra $\Psi_{V,A}(A/\mathrm{Ann}_A(V)) \subset \mathrm{End}_{\mathbb{Z}}(V)$ identified with $A/\mathrm{Ann}_A(V)$ through $\Psi_{V,A}$.

Proposition 1.4. *Suppose that F is a field. Then V is an isotypic left A-module admitting a finite dimensional simple left A-submodule if and only if $A/\mathrm{Ann}_A(V)$ is a finite dimensional simple F-algebra.*

Proof. First, suppose that V is an isotypic left A-module admitting a finite dimensional simple left A-submodule $L \subset V$. Since V is isomorphic in $\mathrm{Vect}(A)$ to the direct sum of non-empty copies of L, we have $\mathrm{Ann}_A(V) = \mathrm{Ann}_A(L)$. By Proposition 1.3, A'_L is a non-zero finite dimensional division F-algebra and $A/\mathrm{Ann}_A(V) = A/\mathrm{Ann}_A(L)$ is isomorphic in $\mathrm{Alg}(F)$ to $\mathrm{M}_n(A'_L)$ with $n \in \mathbb{N} \setminus \{0\}$. Therefore $A/\mathrm{Ann}_A(V)$ is a simple F-algebra with $\dim_F A/\mathrm{Ann}_A(V) = n^2 \dim_F A'_L < \infty$.

Next, suppose that $A/\mathrm{Ann}_A(V)$ is a finite dimensional simple F-algebra. By Wedderburn's theorem (cf. [AF92] 13.4 Theorem), there is a pair of a non-zero finite dimensional division k-algebra D and an isomorphism $\mathrm{M}_n(D) \to A/\mathrm{Ann}_A(V)$ with $n \in \mathbb{N} \setminus \{0\}$ in $\mathrm{Alg}(F)$, and every simple left $A/\mathrm{Ann}_A(V)$-module is isomorphic to $L := A/\mathrm{Ann}_A(V) \otimes_{\mathrm{M}_n(D)} D^n$. Since $A/\mathrm{Ann}_A(V)$ is a simple Artinian ring, it is a semisimple ring by [AF92] 13.5 Proposition. Therefore V is isomorphic in $\mathrm{Vect}(A)$ to the direct sum of non-empty copies of L. By $\dim_F A/\mathrm{Ann}_A(V) \neq 0$, we have $V \neq \{0\}$. Therefore V is an isotypic left A-module admitting a left A-submodule isomorphic to L in $\mathrm{Vect}(A)$. □

A left A-submodule of V is said to be an *isotypic component* if it is a maximal isotypic left A-submodule. Every semisimple left A-module admits a unique direct sum decomposition into isotypic components. We recall the relation between the decomposition of V into the isotypic components and the decomposition of the operator algebra into blocks.

Proposition 1.5. *Suppose that V is semisimple of finite type and the direct sum decomposition $V = \bigoplus_{i=1}^n V_i$ with $n \in \mathbb{N}$ into isotypic components satisfies that $A/\mathrm{Ann}_A(V_i)$ is finitely generated as an F-module for any $i \in \mathbb{N} \cap [1, n]$. Then the direct product $A \to \prod_{i=1}^n A/\mathrm{Ann}_A(V_i)$ of canonical projections induces an isomorphism $A/\mathrm{Ann}_A(V) \to \prod_{i=1}^n A/\mathrm{Ann}_A(V_i)$ in* $\mathrm{Alg}(F)$.

Proof. If $V = \{0\}$, then the assertion holds. Assume $V \neq \{0\}$. Then every isotypic component of V is non-zero. Let $i \in \mathbb{N} \cap [1, n]$. Take a simple left A-module L_i such that V_i is isomorphic in $\mathrm{Vect}(A)$ to the direct product of copies of L_i. By $V_i \neq \{0\}$, we have $\mathrm{Ann}_A(V_i) = \mathrm{Ann}_A(L_i)$. Since L_i is cyclic as a left $A/\mathrm{Ann}_A(L_i)$-module and $A/\mathrm{Ann}_A(L_i) =$

$A/\mathrm{Ann}_A(V_i)$ is finite dimensional, L_i is finite dimensional. Since A'_{L_i} is an F-algebra, L_i is finitely generated as a right A'_{L_i}-module. Therefore $\Psi_{L_i,A}\colon A/\mathrm{Ann}_A(V_i) \hookrightarrow A''_{L_i}$ is an isomorphism in $\mathrm{Alg}(F)$ by Corollary 1.2. Put $L := \bigoplus_{i=1}^n L_i$. Then L admits an injective A-linear homomorphism into V, and V admits an injective A-linear homomorphism into the direct sum of non-empty copies of L. It implies $\mathrm{Ann}_A(L) = \mathrm{Ann}_A(V)$. Since L_i and L_j are simple left A-modules with distinct isomorphisms classes, we have $\mathrm{Hom}_A(L_i, L_j) = \{0\}$ for any $(i, j) \in (\mathbb{N} \cap [1, n])^2$ with $i \neq j$. Therefore the embedding $\bigoplus_{i=1}^n A'_{L_i} \hookrightarrow A'_L$ given as the direct sum of the zero extensions is an isomorphism in $\mathrm{Vect}(F)$. It implies that the map $\iota\colon A''_L \to \prod_{i=1}^n A''_{L_i}$ given as the direct product of the restriction maps is an isomorphism in $\mathrm{Alg}(F)$. Since L is finitely generated as an F-module and A'_L is an F-algebra, L is finitely generated as a right A'_L-module. Therefore $\Psi_{L,A}\colon A/\mathrm{Ann}_A(L) \hookrightarrow A''_L$ is an isomorphism in $\mathrm{Alg}(F)$ by Corollary 1.2. The map $A/\mathrm{Ann}_A(V) \to \prod_{i=1}^n A/\mathrm{Ann}_A(V_i)$ in the assertion coincides with the composite of $(\prod_{i=1}^n \Psi_{L_i,A}^{-1}) \circ \iota \circ \Psi_{L,A}$, and hence is an isomorphism in $\mathrm{Alg}(F)$. □

1.2 Absolute Simplicity and Stable Semisimplicity

Let F be a commutative ring, A an F-algebra, and V a left A-module. We say that V is *absolutely simple* if $K \otimes_F V$ is a simple left $K \otimes_F A$-module for any $K \in \mathrm{ob}(\mathrm{Alg}(F))$ which is a field, is *stably isotypic* if V is isomorphic in $\mathrm{Vect}(A)$ to the direct sum of copies of an absolutely simple left A-module, and is *stably semisimple of finite type* (resp. *stably semisimple*) if V is isomorphic in $\mathrm{Vect}(A)$ to the direct sum of a finite family (resp. a family) of absolutely simple left A-modules.

We say that A is *stably simple* if A is simple and isomorphic in $\mathrm{Alg}(F)$ to $\mathrm{M}_n(F)$ with $n \in \mathbb{N} \setminus \{0\}$, and is *stably semisimple* if A is the direct product of a finite family of stably simple F-algebras. We note that F admits a stably simple F-algebra if and only if F is a field. We recall the structure of an absolutely simple left A-module.

Proposition 1.6. *Suppose that F is a field. Then the following are equivalent:*

(i) The left A-module V is finite dimensional and absolutely simple.

(ii) The left $K \otimes_F A$-module $K \otimes_F V$ is finite dimensional as a K-vector space and is simple for any finite field extension K/F.

(iii) There is a surjective F-algebra homomorphism $\pi\colon A \twoheadrightarrow \mathrm{M}_n(F)$ with $n \in \mathbb{N} \setminus \{0\}$ such that V is isomorphic in $\mathrm{Vect}(A)$ *to* $A/\ker(\pi) \otimes_{\mathrm{M}_n(F)} F^n$.

Proof. The implications from (i) to (ii) and from (iii) to (i) are obvious. We show the implication from (ii) to (iii). Suppose that the left $K \otimes_F A$-module $K \otimes_F V$ is finite dimensional as a K-vector space and simple for any finite field extension K/F. Then $V \cong F \otimes_F V$ is finite dimensional as an F-vector space and simple. Put $D := A'_V$. By Proposition 1.3, there is a surjective F-algebra homomorphism $\pi\colon A \twoheadrightarrow \mathrm{M}_n(D)$ with $n \in \mathbb{N}\setminus\{0\}$ such that V is isomorphic in $\mathrm{Vect}(A)$ to $A/\ker(\pi) \otimes_{\mathrm{M}_n(D)} D^n$, and D is a non-zero finite dimensional division F-algebra. By the existence of a splitting field (cf. [Jac96] Theorem 1.6.19), there is a finite field extension $K/\mathrm{Z}(D)$ such that $K \otimes_{\mathrm{Z}(D)} D$ is isomorphic in Alg(K) to $\mathrm{M}_m(K)$ with $m := (\dim_{\mathrm{Z}(D)} D)^{1/2}$. The inclusion $F \hookrightarrow \mathrm{Z}(D)$ induces a surjective $K \otimes_F A$-linear homomorphism $K \otimes_F V \twoheadrightarrow K \otimes_{\mathrm{Z}(D)} V$, and we have an isomorphism $K \otimes_{\mathrm{Z}(D)} V \cong K \otimes_{\mathrm{Z}(D)} (A/\ker(\pi) \otimes_{\mathrm{M}_n(D)} D^n) \cong (K \otimes_{\mathrm{Z}(A/\ker(\pi))} A/\ker(\pi)) \otimes_{\mathrm{M}_n(\mathrm{M}_m(K))} \mathrm{M}_m(K)^n \cong ((K \otimes_{\mathrm{Z}(A/\ker(\pi))} A/\ker(\pi)) \otimes_{\mathrm{M}_{nm}(K)} K^{nm})^{\oplus m}$ in Vect($K \otimes_F A$). By the assumption, we obtain $m = 1$. It implies $D = \mathrm{Z}(D) = K$. We obtain an isomorphism $K \otimes_F V \cong K \otimes_F (A/\ker(\pi) \otimes_{\mathrm{M}_n(K)} K^n) \cong (K \otimes_F A/\ker(\pi)) \otimes_{\mathrm{M}_n(K \otimes_F K)} (K \otimes_F K)^n$ in Vect($K \otimes_F A$). By the assumption, the kernel N of the multiplication $K \otimes_F K \to K$ is zero, because $(K \otimes_F A/\ker(\pi)) \otimes_{\mathrm{M}_n(K \otimes_F K)} N^n$ is a proper left $K \otimes_F A$-submodule of $(K \otimes_F A/\ker(\pi)) \otimes_{\mathrm{M}_n(K \otimes_F K)} (K \otimes_F K)^n$. It implies that K is isomorphic in Alg(F) to F. We obtain an isomorphism $V \to A/\ker(\pi) \otimes_{\mathrm{M}_n(F)} F^n$ in Vect(A). □

We show the relation between the stably isotypic property of V and the strong simplicity of the operator algebra.

Proposition 1.7. *Suppose that F is a field. Then V is a stably isotypic left A-module admitting a finite dimensional absolutely simple left A-submodule if and only if $A/\mathrm{Ann}_A(V)$ is a stably simple F-algebra.*

Proof. First, suppose that V is a stably isotypic left A-module admitting a finite dimensional absolutely simple left A-submodule L. Since V is isomorphic in Vect(A) to the direct sum of non-empty copies of L, we have $\mathrm{Ann}_A(V) = \mathrm{Ann}_A(L)$. By Proposition 1.6, there is a surjective F-algebra homomorphism $\pi \colon A \twoheadrightarrow \mathrm{M}_n(F)$ with $n \in \mathbb{N} \setminus \{0\}$ such that L is isomorphic in Vect(A) to $A/\ker(\pi) \otimes_{\mathrm{M}_n(F)} F^n$. Therefore $A/\mathrm{Ann}_A(V) = A/\mathrm{Ann}_A(L) = A/\ker(\pi)$ is isomorphic in Alg(F) to $\mathrm{M}_n(F)$.

Next, suppose that $A/\mathrm{Ann}_A(V)$ is a stably simple F-algebra. Take an isomorphism $\mathrm{M}_n(F) \to A/\mathrm{Ann}_A(V)$ with $n \in \mathbb{N} \setminus \{0\}$ in Alg(F). Then $L := A/\mathrm{Ann}_A(V) \otimes_{\mathrm{M}_n(F)} F^n$ is an absolutely simple A-module by Proposition 1.6, and every simple left $A/\mathrm{Ann}_A(V)$-module is isomorphic in Vect(A) to L. Since $A/\mathrm{Ann}_A(V)$ is a simple Artinian ring, it is a semisimple ring by [AF92] 13.5 Proposition. Therefore V is isomorphic in Vect(A) to the direct sum of copies of L. □

By Proposition 1.4, Proposition 1.5, and Proposition 1.7, we obtain the following:

Corollary 1.8. *Suppose that F is a field. If V is semisimple (resp. stably semisimple) of finite type and every simple (resp. absolutely simple) left A-submodule of V is finite dimensional, then $A/\mathrm{Ann}_A(V)$ is a finite dimensional semisimple (resp. stably semisimple) F-algebra.*

We say that A is *discretely spectral* (resp. stably discretely spectral) if A is isomorphic in Alg(F) to the direct product of a family of finite dimensional simple (resp. stably simple) F-algebras. We note that A is finite dimensional and semisimple if and only if A is finite dimensional and discretely spectral by Artin–Wedderburn theorem (cf. [AF92] 13.6 Theorem).

An $e \in \mathrm{Idem}(\mathrm{Z}(A))$ is said to be *primitive* if $e \neq 0$ and $e'e \in \{0, e\}$ for any $e' \in \mathrm{Idem}(\mathrm{Z}(A))$. We denote by $\pi_0(A, F) \subset \mathrm{Idem}(\mathrm{Z}(A))$ the subset of primitive central idempotents. We note that A is discretely spectral if and only if $eA \cong A/(1-e)A$ is a finite dimensional simple F-algebra for any $e \in \pi_0(A, F)$ and the direct product $\Gamma_{A,F} \colon A \to \Gamma(A, F) := \prod_{e \in \pi_0(A,F)} A/(1-e)A$ of canonical projections is bijective. Therefore every discretely spectral F-algebra admits a canonical presentation as the direct product of finite dimensional simple F-algebras. We say that V *vanishes at infinity* if the map $\eta_{V,A,F} \colon V \to \prod_{e \in \pi_0(A,F)} eV$, $v \mapsto (ev)_{e \in \pi_0(A,F)}$ is injective.

The *strong topology on* A is the topology on A generated by the set $\{f + (1-e)A \mid (f, e) \in A \times \pi_0(A, F)\}$, which is Hausdorff if and only if $\ker(\Gamma_{A,F}) = \{0\}$. In particular, every discretely spectral F-algebra is Hausdorff with respect to the strong topology. For an $(f_i)_{i \in I} \in A^I$ with $I \in \mathrm{ob}(\mathrm{Set})$, if the net $(\sum_{i \in S} f_i)_{S \subset I, \# S < \infty}$ indexed by the set of finite subsets of I ordered by inclusions converges to an $f \in A$ with respect to the strong topology, then we say that $\sum^{\mathrm{st}}_{i \in I} f_i$ converges to f, and if such an f is unique, then we write $\sum^{\mathrm{st}}_{i \in I} f_i = f$. We note that $\sum^{\mathrm{st}}_{e \in \pi_0(A,F)} fe$ always converges to f for any $f \in A$, and hence $\pi_0(A, F)$ plays a role of an approximate unit.

Proposition 1.9. *The F-algebra A is discretely spectral (resp. stably discretely spectral) if and only if $A/(1-e)A$ is a finite dimensional simple (resp. stably simple) F-algebra for any $e \in \pi_0(A, F)$, and $\sum^{\mathrm{st}}_{e \in \pi_0(A,F)} f_e e$ uniquely converges for any $(f_e)_{e \in \pi_0(A,F)} \in A^{\pi_0(A,F)}$.*

Proof. The inverse implication follows from the fact that every discretely spectral F-algebra is Hausdorff with respect to the strong topology. Suppose that $A/(1-e)A$ is a finite dimensional simple (resp. stably simple) F-algebra for any $e \in \pi_0(A, F)$, and the sum $\sum^{\mathrm{st}}_{e \in \pi_0(A,F)} f_e e$ uniquely converges for any $(f_e)_{e \in \pi_0(A,F)} \in A^{\pi_0(A,F)}$. Since $\sum_{e \in \pi_0(A,F)} fe$ converges to f for any $f \in A$, the uniqueness of the convergence ensures the injectivity of $\Gamma_{A,F}$. For any $(\overline{f}_e)_{e \in \pi_0(A,F)} \in \Gamma(A, F)$, any lift $(f_e)_{e \in \pi_0(A,F)} \in A^{\pi_0(A,F)}$ of $(\overline{f}_e)_{e \in \pi_0(A,F)}$ satisfies $\Gamma_{A,F}(\sum_{e \in \pi_0(A,F)} f_e e) = (\overline{f}_e)_{e \in \pi_0(A,F)}$. Therefore $\Gamma_{A,F}$ is surjective. Thus A is discretely spectral (resp. stably discretely spectral). □

Let G be a topological monoid. A *smooth F-linear representation of* G is a pair (V, ρ) of a $V \in \mathrm{ob}(\mathrm{Vect}(F))$ and a map $\rho \colon G \times V \to V$ such that the map $\rho(g, -) \colon V \to V, v \mapsto \rho(g, v)$ is an F-linear homomorphism for any $g \in G$, the induced map $\mathrm{BS}(\rho) \colon G \to \mathrm{End}_F(V)$ is a monoid homomorphism with respect to the monoid structure on $\mathrm{End}_F(V)$ given by the composition, and $\{g \in G \mid \rho(g, v) = v'\}$ is open in G for any $(v, v') \in V^2$. An *$F[G]$-linear homomorphism* between smooth F-linear representations of G is an F-linear G-equivariant homomorphism. We denote by $\mathrm{Sm}(F, G)$ the category of smooth F-linear representations of G and $F[G]$-linear homomorphisms.

Let $(V, \rho) \in \mathrm{ob}(\mathrm{Sm}(F, G))$. We say that (V, ρ) is *finite dimensional* if V is finite dimensional, is *irreducible* if V admits exactly two G-stable left F-submodules, is *absolutely irreducible* if $K \otimes_F V$ is an irreducible smooth K-linear representation of G for any $K \in \mathrm{ob}(\mathrm{Alg}(F))$ which is a field, is *isotypic* (resp. *stably isotypic*) if V is isomorphic in $\mathrm{Sm}(F, G)$ to the direct sum of copies of an irreducible (resp. absolutely irreducible) smooth F-linear representation of G, is *semisimple* (resp. *stably semisimple*) *of finite type* if V is isomorphic in $\mathrm{Sm}(F, G)$ to the direct sum of a finite family of isotypic (resp. stably isotypic) smooth F-linear representations of G, is *semisimple* (resp. *stably semisimple*) if V is isomorphic in $\mathrm{Sm}(F, G)$ to the direct sum of a family of irreducible (resp. absolutely irreducible) smooth F-linear representations of G.

Remark 1.10. Let G_0 denote the underlying monoid of G. Then the inclusion $G_0 \hookrightarrow k[G_0]$ induces a fully faithful functor $\phi_{G,F}^{-1} \colon \mathrm{Sm}(G, F) \to \mathrm{Vect}(F[G_0])$ preserving the underlying F-modules, which admits a strict inverse $\phi_{G,F} \colon \mathrm{Vect}(F[G_0]) \to \mathrm{Sm}(G, F)$ if G is a discrete monoid. The smooth F-linear representation (V, ρ) of G is finite dimensional (resp. irreducible, isotypic, semisimple of finite type, semisimple, absolutely irreducible, stably isotypic, stably semisimple of finite type, stably semisimple) if and only if $\phi_{G,F}^{-1}(V)$ is a finite dimensional (resp. simple, isotypic, semisimple of finite type, semisimple, absolutely simple, stably isotypic, stably semisimple of finite type, stably semisimple) left $F[G_0]$-module.

1.3 Banach Representations of Topological Monoids

Let k be a complete valuation field. We recall Banach k-vector spaces and Banach k-linear representations of topological monoids. For a k-vector space V, a *complete non-Archimedean norm on* V is a map $\| - \| \colon V \to [0, \infty)$ is an ultrametric function (cf. [BGR84] 1.1.1 Definition 1) on the underlying Abelian group of V with $\|cv\| = |c| \, \|v\|$ for any $(c, v) \in k \times V$ such that the induced ultrametric (cf. [BGR84] 1.1.3 p/ 12) is complete. A *Banach k-vector space* is a pair $(V, \| - \|)$ of a k-vector space V and a complete non-Archimedean norm $\| - \|$ on V. We abbreviate a Banach k-vector space $(V, \| - \|)$ to V as long as there is no ambiguity of the norm, and equip V with the norm topology so that V forms a topological k-vector space. For example, k itself is a Banach k-vector space. For Banach k-vector spaces V_1 and V_2, a k-linear homomorphism $F \colon V_1 \to V_2$ is said to be *bounded* if there is a $C > 0$ with $\|Fv\| \leq C\|v\|$ for any $v \in V_1$. We denote by $\mathrm{Vect}^{\mathrm{Ban}}(k)$ the category of Banach k-vector spaces and bounded k-linear homomorphisms. We abbreviate $\mathrm{Hom}_{\mathrm{Vect}^{\mathrm{Ban}}(k)}$ (resp. $\mathrm{End}_{\mathrm{Vect}^{\mathrm{Ban}}(k)}$, $\mathrm{Hom}_{\mathrm{Vect}^{\mathrm{Ban}}(k)}(-, k)$) to $\mathrm{Hom}^{\mathrm{cont}}$ (resp. $\mathscr{B}$, $(-)^{\mathbb{D}}$), and equip it with the operator norm (cf. [BGR84] 2.1.6 Definition 2) so that it forms a Banach k-vector space.

For a $(V_1, V_2) \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(k))^2$, an $F \in \mathrm{Hom}^{\mathrm{cont}}(V_1, V_2)$ is said to be *submetric* (resp. *isometric*) if the inequality $\|Fv\| \leq \|v\|$ (resp. the equality $\|Fv\| = \|v\|$) holds for any $v \in V_1$. We denote by $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k) \subset \mathrm{Vect}^{\mathrm{Ban}}(k)$ the subcategory of submetric k-linear homomorphisms. We abbreviate $\mathrm{Hom}_{\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k)}$ (resp. $\mathrm{End}_{\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k(1))}$) to $\mathrm{Hom}_{\leq 1}$ (resp. $\mathscr{B}_{\leq 1}$). We note that a morphism in $\mathrm{Vect}^{\mathrm{Ban}}(k)$ is an isomorphism in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k)$ if and only if it is an isometric bijective map. Let $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k))$. We put $V(1) := \{v \in V \mid \|v\| \leq 1\}$, $V(1-) := \{v \in V(1) \mid \|v\| < 1\}$, and by $\overline{V} := V(1)/V(1-)$. In particular, $k(1)$ is the valuation ring of k, $k(-1)$ is the maximal ideal of $k(1)$, and $\overline{k}$ is the residue field of $k(1)$. The correspondence $V \rightsquigarrow V(1)$ (resp. $V \rightsquigarrow \overline{V}$) gives a faithful functor $(-)(1) \colon \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k) \to \mathrm{Vect}(k(1))$ (resp. a functor $\mathrm{red} \colon \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k) \to \mathrm{Vect}(\overline{k})$). For any closed k-vector subspace $W \subset V$, the map $\overline{W} \to \overline{V}$ induced by the inclusion $W \hookrightarrow V$ is injective, and hence we regard $\overline{W}$ as a $\overline{k}$-vector subspace of $\overline{V}$.

Let $I \in \mathrm{ob}(\mathrm{Set})$. We denote by $\prod_{i \in I}^{\mathrm{Ban}}$ the direct product (cf. the bounded direct product $b(\prod_{i \in I})$ in [BGR84] 2.1.5 Definition 2) of a family in $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$ indexed by I, and by $\widehat{\bigoplus}_{i \in I}$ the direct sum (cf. the restricted direct product $c(\prod_{i \in I})$ in [BGR84] 2.1,5 Definition 3) in $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$ of a family in $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$ indexed by I. We put $\mathrm{C}_0(I, k) := \widehat{\bigoplus}_{i \in I} k$. We recall basic properties of $\mathrm{C}_0(I, k)$.

Proposition 1.11. *The following hold:*

(i) The map $\overline{k}^{\oplus I} \to \overline{\mathrm{C}_0(I, k)}$ *induced by the inclusion* $k(1)^{\oplus I} \hookrightarrow \mathrm{C}_0(I, k)(1)$ *is an isomorphism in* $\mathrm{Vect}(\overline{k})$.

(ii) The map $\mathscr{B}(\mathrm{C}_0(I, k)) \to \prod_{i \in I}^{\mathrm{Ban}} \mathrm{C}_0(I, k)$ *induced by the natural embedding* $I \hookrightarrow \mathrm{C}_0(I, k)(1)$ *is an isomorphism in* $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$.

(iii) The map $\mathrm{End}_{\overline{k}}(\overline{k}^{\oplus I}) \to \prod_{i \in I} \overline{k}^{\oplus I}$ *induced by the natural embedding* $I \hookrightarrow \overline{k}^{\oplus}$ *is an isomorphism in* $\mathrm{Vect}(\overline{k})$.

(iv) The map $\overline{\mathscr{B}(\mathrm{C}_0(I, k))} \to \mathrm{End}_{\overline{k}}(\overline{\mathrm{C}_0(I, k)})$ *induced by the reduction* $\mathrm{C}_0(I, k)(1) \twoheadrightarrow \overline{\mathrm{C}_0(I, k)}$ *is an isomorphism in* $\mathrm{Alg}(\overline{k})$.

(v) The evaluation map $\mathrm{C}_0(I, k) \to \mathrm{C}_0(I, k)^{\mathbb{DD}}$ *is isometric.*

Proof. The assertion (i) follows from the fact that $\{i \in I \mid |f(i)| = 1\}$ is a finite set for any $f \in \mathrm{C}_0(I, k)(1)$. The assertion (ii) follows from the universality of the direct product and the direct sum. The assertion (iii) follows from the assertion (ii) for the case where the valuation of k is trivial. The assertion (iv) follows from the assertions (i), (ii), and (iii) because the direct product commutes with the reduction. The assertion (v) follows from the fact that every $i \in I$ corresponds to the evaluation map $\mathrm{C}_0(I, k) \to k$, $f \mapsto f(i)$. □

We say that V is *finite dimensional* if $\dim_k V < \infty$, is *unramified* if $\|V\| \subset [0, \infty)$ is contained in the closure of $|k|$, and is *orthonormalisable* if V is isomorphic in $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$ to $\mathrm{C}_0(I, k)$ for some $I \in \mathrm{ob}(\mathrm{Set})$. We denote by $\mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(k) \subset \mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$ the full subcategory of unramified Banach k-vector spaces. The functor $(-)(1) \colon \mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k) \to \mathrm{Vect}(k(1))$ induces a fully faithful functor $\mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(k) \to \mathrm{Vect}(k(1))$. We recall basic relations between norms, topologies, endomorphisms, and reductions.

Proposition 1.12. *Let* $W \subset V$ *be a* k*-vector subspace. Then the following hold:*

(i) If W *is finite dimensional, then* W *is closed.*

(ii) If the valuation of k *is discrete and* W *is finite dimensional, then the canonical projection* $V \twoheadrightarrow V/W$ *induces an isomorphism* $\overline{V}/\overline{W} \to \overline{V/W}$ *in* $\mathrm{Vect}(\overline{k})$.

(iii) If V *is unramified and* W *is closed, then* W *and* V/W *are unramified.*

(iv) If the valuation of k is discrete, V is unramified, and W is closed, then the canonical projection $V \twoheadrightarrow V/W$ induces an isomorphism $\overline{V}/\overline{W} \to \overline{V/W}$ in $\mathrm{Vect}(\overline{k})$.

Proposition 1.13. *The following hold:*

(i) If V is orthonormalisable, then V is unramified.

(ii) If the valuation of k is discrete and V is unramified, then V is orthonormalisable, $\|V\| \subset |k|$, *and* $V(1-) = k(1-)V(1)$.

(iii) For a $(V_i)_{i \in I} \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k))^I$ *with* $I \in \mathrm{ob}(\mathrm{Set})$, $\prod^{\mathrm{Ban}}_{i \in I} V_i$ *is unramified if and only if V_i is unramified for any $i \in I$, in which case it satisfies the universality of the direct product in* $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k)$.

(iv) For a $(V_i)_{i \in I} \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k))^I$ *with* $I \in \mathrm{ob}(\mathrm{Set})$, $\widehat{\bigoplus}_{i \in I} V_i$ *is unramified if and only if V_i is unramified for any $i \in I$, in which case it satisfies the universality of the direct sum in* $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k)$.

(v) If V is orthonormalisable and $\overline{V}$ is finite dimensional, then V is finite dimensional and the equality $\dim_k V = \dim_{\overline{k}} \overline{V}$ *holds.*

(vi) If V is finite dimensional, then every k-linear homomorphism $V \to W$ to a $W \in$ $\mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(k))$ *is bounded.*

Proof of Proposition 1.12. The assertion (i) follows from [BGR84] 2.3.3 Proposition 4. The assertion (ii) follows from [BGR84] 2.4.3 Corollary 11. The assertion (iii) follows from the definition of the unramified property. The assertion (iv) follows from [BGR84] 1.1.5 Proposition 4. □

Proof of Proposition 1.13. The assertion (i) follows from the definition of the norm on the direct sum. The assertion (ii) follows from [Mon70] IV 3 Corollaire 1 (cf. [BGR84] 2.5.2 Lemma 2 or the proof of [Sch02] Proposition 10.1) and the equality $\|cv\| = |c| \, \|v\|$ for any $(c, v) \in k \times V$. The assertion (iii) follows from the definition of the unramified property. The assertion (iv) follows from the assertion (iii) and Proposition 1.12 (iii). The assertion (v) follows from Proposition 1.11 (i). The assertion (vi) follows from [BGR84] 2.3.3 Corollary 5. □

A subset $S \subset V$ is said to be *bounded* if there is a $C > 0$ with $\|v\| \leq C$ for any $v \in S$. We have a canonical way to construct an unramified Banach k-vector space from a Banach k-vector space reflecting the boundedness of a multiplicative submonoid of $\mathscr{B}(V)$.

Proposition 1.14. *Suppose that k is a spherically complete field with* $|k| \neq \{0, 1\}$. *Let* $(W, \| - \|) \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k))$ *and* $S \subset \mathscr{B}(W, \| - \|)$. *If* $1 \in S$, $FF' \in S$ *for any* $(F, F') \in S^2$, *and* $\sup_{F \in S} \|F\| < \infty$, *then there is a complete non-Archimedean norm* $\| - \|'$ *on W with* $(W, \| - \|') \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k))$ *and* $\|Fw\|' \leq \|w\|'$ *for any* $F \in S$ *such that the identity* $(W, \| - \|) \to (W, \| - \|')$ *is an isomorphism in* $\mathrm{Vect}^{\mathrm{Ban}}(k)$.

Proof. Let $w \in W$. For any $(F, v) \in S \times (W, \|-\|)^{\mathbb{D}}(1)$, we have $|v(Fw)| \leq \|v\| \, \|F\| \, \|w\| \leq (\sup_{F \in S} \|F\|)\|v\|$. Put $\|w\|' := \sup_{(F,v) \in S \times W^{\mathbb{D}}(1)} \|v(Fw)\| \leq (\sup_{F \in S} \|F\|) \, \|w\|$. By $|k| \neq \{0, 1\}$, there is a $c \in k^{\times}$ with $0 < |c| < 1$. We have $|c| \, \|w\| \leq \sup(|k| \cap [0, \|w\|]) \leq \sup_{v \in W^{\mathbb{D}}(1)} |v(w)| \|w\|$ by Hahn–Banach theorem (cf. [Ing52] Theorem 3 or [Sch02] Corollary 9.3), and $\sup_{v \in W^{\mathbb{D}}(1)} |v(w)| \|w\| \leq \|w\|'$ by $1 \in S$. It implies that $(W, \|-\|')$ is a Banach k-vector space such that the identity $(W, \|-\|) \to (W, \|-\|')$ is an isomorphism in $\mathrm{Vect}^{\mathrm{Ban}}(k)$. By definition, $(W, \|-\|')$ is unramified. Let $(F, w) \in S \times W$. For any $(F', v) \in S \times W^{\mathbb{D}}$, we have $|v(F'(Fw))| = |v((F'F)w)| \leq \|w\|'$. It implies $\|Fw\|' \leq \|w\|'$. □

Applying Proposition 1.14 to the case $S = \mathscr{B}(V)(1)$, we obtain the following:

Corollary 1.15. *Suppose that k is a spherically complete field with $|k| \neq \{0, 1\}$. Let $(W, \|-\|) \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k))$. Then there is a complete non-Archimedean norm $\|-\|'$ on W with $(W, \|-\|') \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k))$ and $\|Fw\|' \leq \|w\|'$ for any $F \in \mathscr{B}(V)(1)$ such that the identity $(W, \|-\|) \to (W, \|-\|')$ is submetric and is an isomorphism in* $\mathrm{Vect}^{\mathrm{Ban}}(k)$.

Proof. By Proposition 1.14, W admits a complete non-Archimedean norm $\|-\|'$ on W with $(W, \|-\|') \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k))$ and $\|Fw\|' \leq \|w\|'$ for any $F \in \mathscr{B}(V)(1)$ such that the identity $(W, \|-\|) \to (W, \|-\|')$ is an isomorphism in $\mathrm{Vect}^{\mathrm{Ban}}(k)$. By $|k| \neq \{0, 1\}$, there is a $c \in k^{\times}$ such that the map $(W, \|-\|) \to (W, \|-\|')$, $w \mapsto cw$ is submetric. By the construction, the map $\|-\|'' \colon W \to [0, \infty)$, $w \mapsto \|cw\|'$ is a complete non-Archimedean norm with $(W, \|-\|'') \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k))$ and $\|Fw\|'' \leq \|w\|''$ for any $F \in \mathscr{B}(V)(1)$ such that the identity $(W, \|-\|) \to (W, \|-\|'')$ is submetric and is an isomorphism in $\mathrm{Vect}^{\mathrm{Ban}}(k)$. □

Let G be a topological monoid. A *Banach k-linear representation of* G is a pair (V, ρ) of a $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(k))$ and a continuous map $\rho \colon G \times V \to V$ such that the map $\rho(g, -) \colon V \to V, v \mapsto \rho(g, v)$ is k-linear for any $g \in G$ and the induced map $\mathrm{BS}(\rho) \colon G \to \mathscr{B}(V)$ is a monoid homomorphism with respect to the monoid structure on $\mathscr{B}(V)$ given by the composition. A *$k[G]$-linear homomorphism* between Banach k-linear representations of G is a k-linear G-equivariant homomorphism. We denote by $\mathrm{Vect}^{\mathrm{Ban}}(G, k)$ the category of Banach k-linear representations of G and bounded $k[G]$-linear homomorphisms.

Let $(V, \rho) \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(G, k))$. We say that (V, ρ) is *finite dimensional* if V is finite dimensional, is *submetric* if $\mathrm{BS}(\rho)$ factors through $\mathscr{B}_{\leq 1}(V) \subset \mathscr{B}(V)$, and is *unitary* if (V, ρ) is submetric and V is unramified. We denote by $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(G, k) \subset \mathrm{Vect}^{\mathrm{Ban}}(G, k)$ the subcategory of submetric Banach k-linear representations of G and submetric $k[G]$-linear homomorphisms, and by $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, k) \subset \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(G, k)$ the full subcategory of unitary Banach k-linear representations of G. For a $(V_i, \rho_i)_{i \in I} \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(G, k))^I$ with $I \in \mathrm{ob}(\mathrm{Set})$, the pair $\prod^{\mathrm{Ban}}_{i \in I} \rho \colon \prod^{\mathrm{Ban}}_{i \in I}(V_i, \rho_i)$ of $\prod^{\mathrm{Ban}}_{i \in I} V_i$ and the entry-wise action $G \times \prod^{\mathrm{Ban}}_{i \in I} V_i \to \prod^{\mathrm{Ban}}_{i \in I} V_i$ of G forms a submetric Banach k-linear representation of G satisfying the universality of the direct product in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k)$, and the closed G-stable k-vector subspace $\widehat{\bigoplus}_{i \in I} V_i \subset \prod^{\mathrm{Ban}}_{i \in I}(V_i, \rho_i)$ forms a submetric Banach k-linear representation of G with respect to the restriction of $\prod^{\mathrm{Ban}}_{i \in I} \rho$ satisfying the universality of the direct sum in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k)$. For any $(V_i, \rho_i)_{i \in I} \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, k))^I$ with $I \in \mathrm{ob}(\mathrm{Set})$, $\prod^{\mathrm{Ban}}_{i \in I}(V_i, \rho_i)$ (resp. $\widehat{\bigoplus}_{i \in I}(V_i, \rho_i)$) forms a unitary Banach k-linear representation of G satisfying the universality of the direct product (resp. the direct sum) in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, k)$ by Proposition 1.13 (iii) and (iv).

We say that (V, ρ) is *irreducible* if (V, ρ) admits exactly two closed G-stable k-vector subspaces, and is *absolutely irreducible* if $K \hat{\otimes}_k V$ (cf. [BGR84] 2.1.7 p. 71) is an irreducible unitary Banach K-linear representation of G with respect to a unique continuous K-linear extension $G \times (K \hat{\otimes}_k V) \to K \hat{\otimes}_k V$ (cf. [BGR84] 2.1.7 Proposition 5) of ρ for any extension K/k of complete valuation fields. By Corollary 1.15, we have the following:

Proposition 1.16. *If that the valuation of k is discrete and (V, ρ) is a submetric Banach k-linear representation of G, then (V, ρ) admits a submetric isomorphism in* $\mathrm{Vect}^{\mathrm{Ban}}(G, k)$ *to a unitary Banach k-linear representation of G.*

Henceforth, suppose that (V, ρ) is submetric. We say that (V, ρ) is *isotypic* (resp. *stably isotypic*) if V admits a submetric injective $k[G]$-linear homomorphism into the direct product of copies of an irreducible (resp. absolutely irreducible) submetric Banach k-linear representation of G, is *semisimple* (resp. *stably semisimple*) *of finite orthogonal type* if V is isomorphic in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, k)$ to the direct sum of a finite family of isotypic (resp. stably isotypic) submetric Banach k-linear representations of G, and is *semisimple* (resp. stably semisimple) if V admits a submetric injective $k[G]$-linear homomorphism into the direct product of a family of irreducible (resp. absolutely irreducible) submetric Banach k-linear representations of G. We denote by $\overline{(V, \rho)}$ the pair of $\overline{V}$ and the map $\overline{\rho} \colon G \times \overline{V} \to \overline{V}$ induced by ρ. Then $\overline{(V, \rho)}$ forms a smooth $\overline{k}$-linear representation of G. The correspondence $(V, \rho) \rightsquigarrow \overline{(V, \rho)}$ gives a functor $\mathrm{red} \colon \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(G, k) \to \mathrm{Sm}(G, \overline{k})$. We note that red does not necessarily preserve the variants of the semisimplicity.

Remark 1.17. Let F be a field. We equip F with the trivial valuation so that F forms a complete valuation field. Then the forgetful functor $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(F) \to \mathrm{Vect}(F)$ is an equivalence of categories, and induces a categorical equivalence $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, F) \to \mathrm{Sm}(G, F)$. The reduction $F = F(1) \twoheadrightarrow \overline{F}$ is an isomorphism in Ring, and $\mathrm{red} \colon \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(G, F) \to \mathrm{Sm}(G, \overline{F})$ is faithful.

Suppose that (V, ρ) is unitary. When the valuation of k is discrete, then the term "submetric" in the definition of the variants of the semisimplicity in the last paragraph can be replaced by "unitary" by Proposition 1.13 (iii) and (iv) and Proposition 1.16. We say that (V, ρ) is *orthogonally stably isotypic* if V is isomorphic in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, k)$ to the direct sum of copies of an absolutely irreducible unitary Banach k-linear representation of G, and is *orthogonally stably semisimple* if V is isomorphic in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(G, k)$ to the direct sum of absolutely simple unitary Banach k-linear representations of G.

2 Banach Modules and Operator Algebras

Let k be a complete valuation field. The aim of this section is to introduce several variants of the semisimplicity of Banach k-algebras and Banach modules over them, and study relations between the semisimplicity of Banach modules and of operator algebras associated to them.

2.1 Banach Modules of Banach Algebras

A *Banach k-algebra* is a pair $(A, \|-\|)$ of a k-algebra A and a complete non-Archimedean norm $\|-\|$ on the underlying k-vector space of A with $\|1\| \in \{0, 1\}$ such that the multiplication $A \times A \to A$ extends to a unique bounded k-linear homomorphism $A \hat{\otimes}_k A \to A$. We abbreviate a Banach k-algebra $(A, \|-\|)$ to A as long as there is no ambiguity of $\|-\|$, and equip A with the norm topology so that A forms a topological k-algebra. We denote by $\mathrm{Alg}^{\mathrm{Ban}}(k)$ the category of Banach k-algebras and bounded k-algebra homomorphisms.

Let $(A, \|-\|) \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}(k))$. We say that $(A, \|-\|)$ is *finite dimensional* if $\dim_k A < \infty$, is *submetric* if $\|ff'\| \leq \|f\| \, \|f'\|$ for any $(f, f') \in A^2$, is *unitary* if $(A, \|-\|)$ is submetric and the underlying Banach k-vector space of A is unramified. For example, k itself is a finite dimensional unitary Banach k-algebra. One of the simplest example of a Banach k-algebra is a closed k-subalgebra of the full operator algebra $\mathscr{B}(V)$ with $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(k))$. By Proposition 1.11 (iv) and Proposition 1.13 (i) and (ii), we obtain the following:

Proposition 2.1. *Let* $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(k))$. *Then every closed k-subalgebra of* $\mathscr{B}(V)$ *is submetric. If the valuation of k is discrete and V is unramified, then every closed k-subalgebra* $B \subset \mathscr{B}(V)$ *is unramified and satisfies* $\mathrm{Ann}_{\overline{B}}(\overline{V}) = \{0\}$.

By the same calculation as the proof of [BGR84] 1.2.4 Proposition 4 and 1.2.4 Corollary 5 for the commutative case, we obtain the following:

Proposition 2.2. *For any closed* $k(1)$*-subalgebra* $B \subset A$, $1 + (B \cap A(1-))$ *is contained in* $B^{\times}$*, and if A is submetric, then* $1 + (B \cap A(1-))$ *forms an open subgroup of* $(B \cap A(1))^{\times}$.

We denote by $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k) \subset \mathrm{Alg}^{\mathrm{Ban}}(k)$ the subcategory of submetric Banach k-algebras and submetric k-algebra homomorphisms, and by $\mathrm{Alg}^{\mathrm{Ban}}_{\mathrm{unit}}(k) \subset \mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$ the full subcategory of unitary Banach k-algebras. The functor $\mathrm{red}\colon \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k) \to \mathrm{Vect}(\overline{k})$ induces a functor $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k) \to \mathrm{Alg}(\overline{k})$. We put $(A, \|-\|)^{\mathrm{op}} := (A^{\mathrm{op}}, \|-\|) \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}(k))$. The correspondence $(A, \|-\|) \rightsquigarrow (A, \|-\|)^{\mathrm{op}}$ gives a functor $(-)^{\mathrm{op}}\colon \mathrm{Alg}^{\mathrm{Ban}}(k) \to \mathrm{Alg}^{\mathrm{Ban}}(k)$. For any $(A_i)_{i \in I} \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k))^I$ (resp. $(A_i)_{i \in I} \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}_{\mathrm{unit}}(k))^I$) with $I \in \mathrm{ob}(\mathrm{Set})$, $\prod^{\mathrm{Ban}}_{i \in I} A_i$ forms a submetric (resp. unitary) Banach k-algebra satisfying the universality of the direct product in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$ (resp. $\mathrm{Alg}^{\mathrm{Ban}}_{\mathrm{unit}}(k)$ by Proposition 1.13 (iii)).

Let $A \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}(k))$. A *Banach left A-module* is a pair $(V, \|-\|)$ of a left A-module V and a complete non-Archimedean norm $\|-\|$ on the underlying k-vector space of V such that the scalar multiplication $A \times V \to V$ extends to a unique bounded k-linear homomorphism $A \hat{\otimes}_k V \to V$. We abbreviate a Banach left A-module $(V, \|-\|)$ to V as long as there is no ambiguity of the norm, and equip V with the norm topology so that V forms a topological left A-module. A *Banach right A-module* is a Banach left A^{op}-module. We denote by $\mathrm{Vect}^{\mathrm{Ban}}(A)$ the category of Banach left A-modules and bounded A-linear homomorphisms. We abbreviate $\mathrm{Hom}_{\mathrm{Vect}^{\mathrm{Ban}}(A)}$ (resp. $\mathrm{End}_{\mathrm{Vect}^{\mathrm{Ban}}(A)}$) to $\mathrm{Hom}^{\mathrm{cont}}_A$ (resp. $\mathrm{End}^{\mathrm{cont}}_A$), and equip it with the restriction of the operator norm so that it forms a Banach k-vector space. One of the simplest example of a Banach left A-module is a closed left ideal of A. Let $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(A))$.

Proposition 2.3. *The two-sided ideal* $\mathrm{Ann}_A(V) \subset A$ *and the left ideal* $\mathrm{Ann}_A(v) \subset A$ *are closed for any* $v \in V$.

Proof. For any $v \in V$, $\mathrm{Ann}_A(v)$ coincides with the kernel of the map $A \to V$, $f \mapsto fv$, which is closed because the scalar multiplication $A \times V \to V$ is continuous and V is Hausdorff. We have $\mathrm{Ann}_A(V) = \bigcap_{v \in V} \mathrm{Ann}_A(v)$, and hence $\mathrm{Ann}_A(V)$ is closed. □

We say that V is *finite dimensional* if $\dim_k V < \infty$, is *submetric* if the map $\Pi_{V,A} \colon A \to \mathscr{B}(V)$ induced by the scalar multiplication $A \times V \to V$ is submetric, and is said to be *unitary* if V is submetric and the underlying Banach k-vector space of V is unramified. We denote by $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A) \subset \mathrm{Vect}^{\mathrm{Ban}}(A)$ the subcategory of submetric Banach left A-modules and submetric A-linear homomorphisms, and by $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(A) \subset \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A)$ the full subcategory of unitary Banach left A-modules. The functor red: $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k) \to \mathrm{Vect}(\overline{k})$ induces a functor $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A) \to \mathrm{Vect}(\overline{A})$, $V \rightsquigarrow \overline{V}$. One of the simplest example of a submetric Banach left module is the natural representation of an operator algebra.

Proposition 2.4. *Let* $W \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(k))$. *For any closed k-subalgebra* $B \subset \mathscr{B}(W)$, *W forms a submetric Banach left B-module with* $\mathrm{Ann}_B(W)\{0\}$.

We say that V is *simple* if V admits exactly two closed left A-submodules, and is *absolutely simple* if $K \hat{\otimes}_k V$ is a simple Banach left $K \hat{\otimes}_k A$-module for any extension K/k of complete valuation fields. By Corollary 1.15, we have the following:

Proposition 2.5. *If that the valuation of k is discrete and V is a submetric Banach left A-module, then V admits a submetric isomorphism in* $\mathrm{Vect}^{\mathrm{Ban}}(A)$ *to a unitary Banach left A-module.*

We say that A is *simple* if A admits exactly two closed two-sided ideal, is *stably simple* (resp. *orthogonally stably simple*) if A is isomorphic in $\mathrm{Alg}^{\mathrm{Ban}}(k)$ (resp. $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$) to $\mathrm{M}_n(k)$ with $n \in \mathbb{N} \setminus \{0\}$. We note that every non-zero ring admits a maximal left ideal and a maximal two-sided ideal by a standard argument with Zorn's lemma. Therefore the following ensures the existence of a simple Banach left A-module:

Proposition 2.6. *Every maximal left (resp. two-sided) ideal* $\wp \subset A$ *is closed, and* $A/\wp$ *forms a simple Banach left A-module (resp. a simple Banach k-algebra).*

Proof. The first assertion immediately follows from Proposition 2.2 for $B = A(1)$, and the second assertion follows from the fact that the underlying left A-module (resp. the underlying k-algebra) of $A/\wp$ is simple. □

Henceforth, suppose that A is submetric. For an $I \in \mathrm{ob}(\mathrm{Set})$, we denote also by $\prod^{\mathrm{Ban}}_{i \in I}$ the direct product (cf. the bounded direct product $b(\prod_{i \in I})$ in [BGR84] 2.1.5 Definition 2) of a family in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A)$ indexed by I, and by $\widehat{\bigoplus}_{i \in I}$ the direct sum (cf. the restricted direct product $c(\prod_{i \in I})$ in [BGR84] 2.1,5 Definition 3) of a family in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A)$ indexed by I. The forgetful functor $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A) \to \mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k)$ strictly commutes with the direct product and the direct sum. Therefore for a $(V_i)_{i \in I} \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A))^I$ with $I \in \mathrm{ob}(\mathrm{Set})$,

$\prod_{i\in I}^{\mathrm{Ban}} V_i$ (resp. $\widehat{\bigoplus}_{i\in I} V_i$) is unitary if and only if V_i is unramified for any $i \in I$, in which case it satisfies the universality of the direct product (resp. the direct sum) in $\mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(A)$ by Proposition 1.13 (iii) (resp. (iv)).

Suppose that V is submetric. We say that V is *isotypic* (resp. *stably isotypic*) if V admits a submetric injective A-linear homomorphism into the direct product of copies of a simple (resp. an absolutely simple) submetric Banach left A-module, is *semisimple* (resp. *stably semisimple*) *of finite orthogonal type* if V is isomorphic in $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(A)$ to the direct sum of a finite family of isotypic (resp. stably isotypic) submetric Banach left A-modules, and is *semisimple* (resp. *stably semisimple*) if V admits a submetric injective A-linear homomorphism into the direct product of a family of simple (resp. absolutely simple) submetric Banach left A-modules. For example, V is finite dimensional and simple (resp. semisimple) if and only if the underlying left A-module of V is finite dimensional and simple (resp. semisimple) by Proposition 1.12 (i). We study a structure of an isotypic submetric Banach left A-module.

Proposition 2.7. *Suppose that* $|k| \neq \{0, 1\}$ *and* V *is an isotypic submetric Banach left* A*-module admitting a finite dimensional simple Banach left* A*-submodule* L*. Then the following hold:*

(i) *Every simple Banach left* A*-submodule of* V *is isomorphic in* $\mathrm{Vect}^{\mathrm{Ban}}(A)$ *to* L*.*

(ii) *The evaluation map* $V \to L^{\mathrm{Hom}_A^{\mathrm{cont}}(V,L)(1)}$, $v \mapsto (\pi(v))_{\pi\in\mathrm{Hom}_A^{\mathrm{cont}}(V,L)(1)}$ *induces a submetric injective* A*-linear homomorphism* $\hat{\eta}_{V,L,A}\colon V \to \prod_{\pi\in\mathrm{Hom}_A^{\mathrm{cont}}(V,L)(1)}^{\mathrm{Ban}} L$.

(iii) *If* V *is stably isotypic, then* L *is absolutely simple.*

Proof. First, we verify the assertion (ii). We have $\|\pi(v)\| \leq \|v\|$ for any $(v, \pi) \in V \times \mathrm{Hom}_A^{\mathrm{cont}}(V, L)(1)$, and hence the evaluation map induces a submetric A-linear homomorphism $\hat{\eta}_{V,L,A}\colon V \to \prod_{\pi\in\mathrm{Hom}_A^{\mathrm{cont}}(V,L)(1)}^{\mathrm{Ban}} L$. We show the injectivity of $\hat{\eta}_{V,L,A}$. Take a simple unitary Banach left A-module L_0 and an injective submetric A-linear homomorphism $\iota\colon V \hookrightarrow L_0^P$ with $P \in \mathrm{ob}(\mathrm{Set})$. By $L \neq \{0\}$, there is a $p_0 \in P$ such that the composite $\varphi\colon L \to L_0$ of $\iota|_L$ and the p_0-th projection $L_0^P \to L_0$ is non-zero. By Proposition 1.12 (i) and Proposition 1.13 (vi), the simplicity of L and L_0 ensures that φ is an isomorphism in $\mathrm{Vect}^{\mathrm{Ban}}(A)$. Let $v \in V \setminus \{0\}$. By $v \neq 0$, there is a $p \in P$ such that the image of v by the composite $\pi\colon V \to L_0$ of ι and the p-th projection $L_0^P \twoheadrightarrow L_0$ is non-zero. By $|k| \neq \{0, 1\}$, there is a $c \in k^{\times}$ such that $c\varphi^{-1} \circ \pi$ is submetric. We obtain $(c\varphi^{-1} \circ \pi)(v) = c\varphi^{-1}(\pi(v)) \neq 0$. Therefore $\hat{\eta}_{V,L,A}$ is injective.

Next, we verify the assertion (i). Let $L_1 \subset V$ be a simple Banach left A-submodule. By $L_1 \neq \{0\}$, the injectivity of $\hat{\eta}_{V,L,A}$ ensures that there is a $\pi \in \mathrm{Hom}_A^{\mathrm{cont}}(V, L)(1)$ such that $\pi|_{L_1}$ is non-zero. By Proposition 1.12 (i) and the simplicity of L_1 and L, $\pi|_{L_1}$ is bijective, and hence is an isomorphism in $\mathrm{Vect}^{\mathrm{Ban}}(A)$ by Proposition 1.13 (vi). Finally, we verify the assertion (iii). If V is stably isotypic, then L_0 can be chosen to be absolutely simple. Since φ is an isomorphism, L is also absolutely simple. □

Suppose that V is unitary. When the valuation of k is discrete, then the term "submetric" in the definition of the variants of the semisimplicity in the last paragraph can be replaced by "unitary" by Proposition 1.13 (iii) and (iv) and Proposition 2.5. We say that V is *orthogonally stably isotypic* if V is isomorphic in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(A)$ to the direct sum of copies of an absolutely simple unitary Banach left A-module, and is *orthogonally stably semisimple* if V is isomorphic in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(A)$ to the direct sum of absolutely simple unitary Banach left A-modules.

We say that A is *semisimple* (resp. *stably semisimple*, *orthogonally stably semisimple*) if A is isomorphic in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$ to the direct product of a finite family of simple (resp. stably simple, orthogonally stably simple) submetric Banach k-algebras, and is *discretely spectral* (resp. *stably discretely spectral*, *orthogonally stably discretely spectral*) if A is isomorphic in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$ to the direct product of a family of finite dimensional simple (resp. stably simple, orthogonally stably simple) submetric Banach k-algebras. For any $E \in \pi_0(A(1), k(1))$, $(1 - E)A$ coincides with $\mathrm{Ann}_A(EA)$, which is a closed two-sided ideal by Proposition 2.3, and hence the canonical projection $A \twoheadrightarrow A/(1-E)A$ is a morphism in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$. We denote by $\hat{\Gamma}_{A,k} \colon A \to \hat{\Gamma}(A,k) := \prod^{\mathrm{Ban}}_{E \in \pi_0(A(1),k(1))} A/(1-E)A$ the morphism in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$ given as the direct product of canonical projections. The submetric Banach k-algebra A is discretely spectral if and only if $\hat{\Gamma}_{A,k}$ is an isomorphism in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$. When V is submetric, then we put $\hat{\Gamma}(V,A,k) := \prod^{\mathrm{Ban}}_{E \in \pi_0(A(1),k(1))} EV$. The decomposition of A yields a decomposition of a Banach left A-submodules in the following sense:

Proposition 2.8. *The following hold:*

(i) For any $E \in \pi_0(A(1), k(1))$, $EV \subset V$ is a closed left A-submodule.

(ii) If V is submetric, then the map $V \to \prod_{E \in \pi_0(A(1),k(1))} EV$, $v \mapsto (Ev)_{E \in \pi_0(A(1),k(1))}$ induces a submetric A-linear homomorphism $\hat{\Gamma}_{V,A,k} \colon V \to \hat{\Gamma}(V,A,k)$.

Proof. For any $E \in \pi_0(A(1), k(1))$, EV coincides with the kernel of the map $V \to V$, $v \mapsto (1-E)v$, which is closed by the continuity of the scalar multiplication $A \times V \to V$ and the Hausdorff property of V, and the map $V \to EV$, $v \mapsto Ev$ is submetric because V is submetric. We obtain a submetric A-linear homomorphism $V \to \prod^{\mathrm{Ban}}_{E \in \pi_0(A(1),k(1))} EV$ by the universality of the direct product in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(A)$. □

Suppose that V is submetric. We say that V *vanishes at infinity* if $\hat{\Gamma}_{V,A,k}$ is injective. We note that V does not necessarily vanish at infinity even if A is discretely spectral. For example, $\prod^{\mathrm{Ban}}_{i \in \mathbb{N}} k$ is a discretely spectral unitary Banach k-algebra and $(\prod^{\mathrm{Ban}}_{i \in \mathbb{N}} k)/(\widehat{\bigoplus}_{i \in \mathbb{N}} k)$ is a unitary Banach left $\prod^{\mathrm{Ban}}_{i \in \mathbb{N}} k$-module which does not vanish at infinity.

2.2 Semisimplicity of Operator Algebras

Henceforth, suppose that the valuation of k is discrete. Let $A \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}(k))$ and $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}(A))$. We denote by $\mathrm{C}^*(A, V) \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}(k))$ the closure of $\Pi_{V,A}(A) \subset \mathscr{B}(V)$.

Then $\mathrm{C}^*(A, V)$ is submetric by Proposition 2.1, and V is a submetric Banach left $\mathrm{C}^*(A, V)$-module by Proposition 2.4. We study a relation between the semisimplicity of the representation module V and the operator algebra $\mathrm{C}^*(A, V)$. To begin with, we show a relation between the isotypic property of V and the simplicity of $\mathrm{C}^*(A, V)$ as an analogue of Proposition 1.4.

Proposition 2.9. *Suppose that A and V are submetric. Then V is an isotypic (resp. a stably isotypic) submetric Banach left A-module admitting a finite dimensional simple (resp. absolutely simple) Banach left A-submodule if and only if* $\mathrm{C}^*(A, V)$ *is a finite dimensional simple (resp. stably simple) submetric Banach k-algebra.*

In order to verify Proposition 2.9, we study the operator algebra of a simple Banach left A-module.

Lemma 2.10. *If V is a finite dimensional simple (resp. absolutely simple) Banach left A-module, then* $\mathrm{C}^*(A, V)$ *is a finite dimensional simple (resp. stably simple) submetric Banach k-algebra. In addition, if V is a finite dimensional absolutely simple unitary Banach left A-module, then* $\mathrm{C}^*(A, V)$ *is a finite dimensional orthogonally stably simple Banach k-algebra, and V is isomorphic in* $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(A)$ *to* $A \hat{\otimes}_{\mathrm{M}_n(k)} k^n$ *with* $n = \dim_k V$ *for an isomorphism* $\mathrm{M}_n(k) \to A$ *in* $\mathrm{Alg}^{\mathrm{Ban}}_{\mathrm{unit}}(k)$.

Proof. Put $B := \mathrm{C}^*(A, V)$. Suppose that V is a finite dimensional simple Banach left A-module. Put $n := \dim_k V$. We have $\dim_k B \leq n^2 < \infty$. By Proposition 1.12 (i), the underlying left A-module of V is simple, and B coincides with $\Pi_{V,A}(A)$. Therefore by Proposition 1.4, the underlying k-algebra of B is simple. It implies that B is simple. In addition, suppose that V is absolutely simple. By Proposition 1.12 (i), the underlying left $K \otimes_k A$-module of $K \hat{\otimes}_k V$ is isomorphic to $K \otimes_k V$ for any finite field extension K/k, and hence the underlying left A-module of V is absolutely simple by Proposition 1.6. Therefore B coincides with $\mathscr{B}(V)$ again by Proposition 1.6. Therefore a k-linear basis of V yields an isomorphism $B \to \mathrm{M}_n(k)$ in $\mathrm{Alg}^{\mathrm{Ban}}(k)$ by Proposition 1.13 (vi). In addition, if V is unitary, then the underlying Banach k-vector space of V is orthonormalisable by Proposition 1.13 (iii), and an isomorphism $\mathrm{C}_0(\mathbb{N} \cap [1, n], k) \to V$ in $\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(k)$ induces an isomorphism $B \to \mathscr{B}(\mathrm{C}_0(\mathbb{N} \cap [1, n], k)) = \mathrm{M}_n(k)$ in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$. □

Lemma 2.11. *If V is an isotypic submetric Banach left A-module admitting a finite dimensional simple Banach left A-submodule L, then the restriction map* $\Pi_{V,A}(A) \twoheadrightarrow \Pi_{L,A}(A)$ *induces an isomorphism* $\mathrm{C}^*(A, V) \to \mathrm{C}^*(A, L)$ *in* $\mathrm{Alg}^{\mathrm{Ban}}(k)$.

Proof. By Proposition 1.12 (i), the underlying left A-module of L is simple, and $\mathrm{C}^*(A, L)$ coincides with $\Pi_{L,A}(A)$. By Proposition 2.7 (ii), $\hat{\eta}_{V,L,A} \colon V \to \prod_{\pi \in \mathrm{Hom}^{\mathrm{cont}}_A(V,L)(1)} L$ is a submetric injective A-linear homomorphism. Therefore $\Pi_{V,A}$ factors through $A/\mathrm{Ann}_A(L)$, and the restriction map $\Pi_{V,A}(A) \twoheadrightarrow \Pi_{L,A}(A)$ induces an isomorphism $\mathrm{C}^*(A, V) \to \mathrm{C}^*(A, L)$ in $\mathrm{Alg}^{\mathrm{Ban}}(k)$ by Proposition 1.12 (i) and Proposition 1.13 (vi). □

Proof of Proposition 2.9. Put $B := \mathrm{C}^*(A, V)$. Since V is submetric, B is submetric as a Banach left A-module. First, suppose that B is a finite dimensional simple submetric

Banach k-algebra. By Proposition 1.12 (i), B coincides with $\Pi_{V,A}(A)$ and the underlying k-algebra of B is simple. By Wedderburn's theorem (cf. [AF92] 13.4 Theorem), there is an isomorphism $\mathrm{M}_n(D) \to B$ with $n \in \mathbb{N} \setminus \{0\}$ in $\mathrm{Alg}(k)$ for some division k-algebra D, and every simple left B-module is isomorphic to $L := B \otimes_{\mathrm{M}_n(D)} D^n$. Take a $\xi \in L \setminus \{0\}$. By Proposition 2.6, $\mathrm{Ann}_B(\xi)$ is closed and L forms a simple submetric Banach left A-module with respect to the quotient norm associated to the surjective map $B \twoheadrightarrow L,\ f \mapsto f\xi$. We show that V admits a submetric injective A-linear homomorphism into the direct product of copies of L.

We regard $V^{\mathbb{D}}$ as a Banach right A-module in a natural way. Let $w \in V^{\mathbb{D}}$. Then $wA = wB$ is closed by Proposition 1.12 (i). We regard $(wA)^{\mathbb{D}}$ as a Banach left A-module. Put $P_w := \mathrm{Hom}_A^{\mathrm{cont}}((wA)^{\mathbb{D}}, L)(1)$. We denote by $\iota_w \colon V \to (wA)^{\mathbb{D}}$ the composite of the evaluation map $V \to V^{\mathbb{D}\mathbb{D}}$ and the restriction map $V^{\mathbb{D}\mathbb{D}} \to (wA)^{\mathbb{D}}$. Put $P := \sqcup_{w \in V^{\mathbb{D}}} \{w\} \times P_w$. Let $v \in V$. We have $\|\pi(\iota_w(v))\| \leq \|\iota_w(v)\| \leq \|v\|$ for any $(w,\pi) \in P$, and hence $(\pi(\iota_w(v)))_{(w,\pi) \in P} \in \prod^{\mathrm{Ban}}_{(w,\pi) \in P} L$. We denote by $\iota \colon V \to \prod^{\mathrm{Ban}}_{(w,\pi) \in P} L$ the submetric A-linear homomorphism given by setting $\iota(v) := (\pi(\iota_w(v)))_{(w,\pi) \in P}$ for a $v \in V$. We show that ι is injective. Let $v \in V \setminus \{0\}$. Since the valuation of k is discrete, there is a $w \in V^{\mathbb{D}}(1)$ with $w(v) \neq 0$ by Hahn–Banach theorem (cf. [Ing52] Theorem 3 or [Sch02] Corollary 9.3). It implies $\iota_w(v) \neq 0$. By $\dim_k (wA)^{\mathbb{D}} \leq \dim_k B < \infty$, the underlying left A-module of $(wA)^{\mathbb{D}}$ is isomorphic in $\mathrm{Vect}(A)$ to the direct sum of non-empty copies of the underlying left A-module of L. Take a family $(L_i)_{i=1}^m$ with $m \in \mathbb{N} \setminus \{0\}$ of left A-submodules of $(wA)^{\mathbb{D}}$ isomorphic in $\mathrm{Vect}(A)$ to L with $(wA)^{\mathbb{D}} = \bigoplus_{i=1}^m L_i$. By $\iota_w(v) \neq 0$, there is an $i_0 \in \mathbb{N} \cap [1,m]$ with $\iota_w(v) \notin L_{i_0}$. By Proposition 1.12 (i), $L_{i_0}^{\perp} := \bigoplus_{i \in (\mathbb{N} \cap [1,m]) \setminus i_0} L_i \subset (wA)^{\mathbb{D}}$ is closed. Take an isomorphism $\varphi \colon (wA)^{\mathbb{D}}/L_{i_0}^{\perp} \to L$ in $\mathrm{Vect}(A)$. By Proposition 1.13 (vi), replacing φ to $\varpi_k^r \varphi$ for a sufficiently large $r \in \mathbb{N}$, we may assume that φ is submetric. We denote by $\pi \colon (wA)^{\mathbb{D}} \to L$ the composite of the canonical projection $(wA)^{\mathbb{D}} \twoheadrightarrow (wA)^{\mathbb{D}}/L_{i_0}^{\perp}$ and φ. We obtain $(w,\pi) \in P$ and $\pi(\iota_w(v)) \neq 0$. It implies that ι is injective. Therefore V is isotypic. In addition, suppose that $\overline{V}$ is stably isotypic. Then D can be chosen to be k by Proposition 1.4 and Proposition 3.1, and $L = B \otimes_{\mathrm{M}_n(k)} k^n$ is absolutely simple. As a consequence, V is stably isotypic.

Next, suppose that V is an isotypic submetric Banach left A-module admitting a finite dimensional simple Banach left A-submodule $L \subset V$. By Lemma 2.11, we have an isomorphism $B \to \mathrm{C}^*(A, L)$. Therefore B is a finite dimensional simple submetric Banach k-algebra by Lemma 2.10. In addition, suppose that V is stably isotypic. Then L is absolutely simple by Proposition 2.7 (iii). Since B is isomorphic in $\mathrm{Alg}^{\mathrm{Ban}}(k)$ to $\mathrm{C}^*(A, L)$, B is stably simple by Lemma 2.10. □

Henceforth, suppose that V is submetric. In order to apply Proposition 2.9 to criteria of the finite orthogonal type property of V, we prepare the following:

Proposition 2.12. *Let $E \in \pi_0(A(1), k(1))$. If A is discretely spectral (resp. stably discretely spectral), then $EV \subset V$ is an isotypic (resp. a stably isotypic) submetric Banach left A-submodule, and if $EV \neq \{0\}$, then EV admits a finite dimensional simple (resp. absolutely simple) Banach left A-submodule for any $E \in \pi_0(A(1), k(1))$.*

Proof. Suppose that A is discretely spectral (resp. stably discretely spectral). By Proposition 2.8 (i), EV is a closed left A-submodule. Since A is discretely spectral, $A/(1-E)A$ is a finite dimensional simple (resp. stably simple) submetric Banach k-algebra. Therefore $\Pi_{EV,A} \colon A/(1-E)A \to \mathrm{C}^*(A, EV)$ is the zero map or an isomorphism in $\mathrm{Alg}^{\mathrm{Ban}}(k)$ by Proposition 1.12 (i). It implies that EV is zero or an isotypic (resp. a stably isotypic) submetric Banach left A-module admitting a finite dimensional simple (resp. absolutely simple) Banach left A-submodule by Proposition 2.9. □

We obtain a relation between the semisimplicity of $\mathrm{C}^*(A, V)$ and the finite orthogonal type property of V.

Corollary 2.13. *If* $\mathrm{C}^*(A, V)$ *is a finite dimensional semisimple (resp. an orthogonally stably semisimple) submetric Banach k-algebra, then V is a semisimple (resp. stably semisimple) submetric Banach left A-module of finite orthogonal type such that every simple (resp. absolutely simple) Banach left A-submodule is finite dimensional.*

Proof. Put $B := \mathrm{C}^*(A, V)$. By Proposition 2.12, $EV \subset V$ is an isotypic (resp. a stably isotypic) submetric Banach left B-module admitting a finite dimensional simple (resp. absolutely simple) Banach left B-submodule for any $E \in \pi_0(B(1), k(1))$. Since $\mathrm{C}^*(A, V)$ is semisimple, we have $\#\pi_0(B(1), k(1)) < \infty$ and $\sum_{E \in \pi_0(B(1),k(1))} E = 1$. Therefore the finite decomposition $B(1) = \bigoplus_{E \in \pi_0(B(1),k(1))} EB(1)$ into two-sided ideals gives a finite orthogonal decomposition $V = \widehat{\bigoplus}_{E \in \pi_0(B(1),k(1))} EV$ as a Banach left B-module. Let $E \in \pi_0(B(1), k(1))$. Take a finite dimensional simple (resp. absolutely simple) Banach left B-submodule $L \subset EV$. By Lemma 2.11, the restriction map $\Pi_{EV}(A) \twoheadrightarrow \Pi_{L,A}(A)$ induces an isomorphism $B \to \mathrm{C}^*(A, L)$ in $\mathrm{Alg}^{\mathrm{Ban}}(k)$. Therefore L is simple as a Banach left A-module. Since $\hat{\eta}_{EV,L,B}$ is A-linear, EV is isotypic (resp. stably isotypic) as a submetric Banach left A-module by Proposition 2.7 (ii). Therefore V is semisimple (resp. stably semisimple) of finite orthogonal type. Let $L_0 \subset V$ be a simple (resp. an absolutely simple) left A-submodule. Take a $v \in L \setminus \{0\}$. By the presentation $V = \widehat{\bigoplus}_{E \in \pi_0(B(1),k(1))} EV$, there is an $E \in \pi_0(B(1), k(1))$ with $Ev \neq 0$. Since $EL = EV \cap L$ is a closed left A-submodule of L with $Ev \in EL$, we obtain $L = EL \subset EV$. Therefore L is finite dimensional by Proposition 1.12, Proposition 2.7, and Proposition 2.9. □

We introduce a bigger operator algebra $\mathrm{W}^*(A, V)$. The *weak operator topology on* $\mathscr{B}(V)$ is the topology on $\mathscr{B}(V)$ generated by the set $\{\{f' \in \mathscr{B}(V) \mid |w((f'-f)v)| < \epsilon\} \mid (f, v, w, \epsilon) \in \mathscr{B}(V) \times V \times V^{\mathbb{D}} \times (0, \infty)\}$. We denote by $\mathrm{W}^*(A, V) \subset \mathscr{B}(V)$ the closure of $\Pi_{V,A}(A)$ with respect to the weak operator topology. Since the weak operator topology is weaker than or equal to the norm topology, $\mathrm{W}^*(A, V)$ is closed with respect to the norm topology and contains $\mathrm{C}^*(A, V)$. By the continuity of the multiplication $\mathscr{B}(V) \times \mathscr{B}(V) \to \mathscr{B}(V)$ with respect to the norm topology, $\mathrm{W}^*(A, V)$ is contained in the double commutant $\mathrm{End}^{\mathrm{cont}}_{\mathrm{End}^{\mathrm{cont}}_A(V)}(V)$. By Proposition 2.1, $\mathrm{W}^*(A, V)$ is a submetric Banach k-algebra, and V is a submetric Banach left $\mathrm{W}^*(A, V)$-module by Proposition 2.4. We show a relation between the semisimplicity of the representation module V and the discretely spectral property of the operator algebra $\mathrm{W}^*(A, V)$.

Proposition 2.14. *If* $\mathrm{W}^*(A, V)$ *is discretely spectral (resp. stably discretely spectral) and* V *vanishes at infinity as a submetric Banach left* $\mathrm{W}^*(A, V)$*-module, then* V *is a semisimple (resp. stably semisimple) submetric Banach left* A*-module.*

In order to verify Proposition 2.14, we compare the isotypic property of a closed $\mathrm{W}^*(A, V)$-submodule and that of its underlying Banach left A-submodule.

Lemma 2.15. *If* $\mathrm{W}^*(A, V)$ *is discretely spectral (resp. stably discretely spectral) and* V *vanishes at infinity as a submetric Banach left* $\mathrm{W}^*(A, V)$*-module, then every simple (resp. absolutely simple) Banach left* $\mathrm{W}^*(A, V)$*-submodule of* V *is a finite dimensional simple (resp. absolutely simple) Banach left* A*-module, and every isotypic (resp. stably isotypic) submetric Banach left* $\mathrm{W}^*(A, V)$*-submodule of* V *is an isotypic (resp. a stably isotypic) submetric Banach left* A*-module.*

Proof. Put $B := \mathrm{W}^*(A, V)$. Suppose that $\mathrm{W}^*(A, V)$ is discretely spectral (resp. stably discretely spectral) and V vanishes at infinity as a Banach left $\mathrm{W}^*(A, V)$-module. Let $L \subset V$ be a simple (resp. an absolutely simple) Banach left $\mathrm{W}^*(A, V)$-submodule. By the injectivity of $\hat{\Gamma}_{V,A,k}$, there is an $E \in \pi_0(B(1), k(1))$ with $EL = L$. Since $\mathrm{W}^*(A, V)/(1 - E)\mathrm{W}^*(A, V)$ is finite dimensional, so is L by Proposition 1.12 (i). We have $\Pi_{L,B}(B) = \Pi_{L,A}(A) = \mathrm{C}^*(A, V)$ again by Proposition 1.12 (i). Therefore L is simple (resp. absolutely simple) as a submetric Banach left A-module.

Let $W \subset V$ be an isotypic (resp. a stably isotypic) submetric Banach left B-submodule of V. If $W = \{0\}$, then W is a stably isotypic submetric Banach left A-module. Assume $W \neq \{0\}$. By the injectivity of $\hat{\Gamma}_{V,A,k}$, there is an $E \in \pi_0(B(1), k(1))$ with $EW \neq \{0\}$. By Proposition 2.12, EW is an isotypic submetric Banach left B-module admitting a simple (resp. an absolutely simple) Banach left B-submodule L. In particular, we have $(1 - E)L = \{0\}$. By Proposition 2.7, we obtain $(1 - E)W = \{0\}$. It implies $EW = W$. By the argument above, L is a simple (resp. an absolutely simple) submetric Banach left A-module. Since $\hat{\eta}_{W,L,B}$ is A-linear, W is an isotypic (resp. a stably isotypic) submetric Banach left A-module by Proposition 2.7 (ii). □

Proof of Proposition 2.14. Put $B := \mathrm{W}^*(A, V)$. By Proposition 2.12 and Lemma 2.15, $EV \subset V$ is an isotypic (resp. a stably isotypic) submetric Banach left A-submodule for any $E \in \pi_0(B(1), k(1))$. By the injectivity of $\hat{\Gamma}_{V,A,k}$, V is semisimple (resp. stably semisimple). We verified that every simple (resp. absolutely simple) left A-submodule is finite dimensional in the proof of Lemma 2.15. □

3 Semisimplicity and Reduction

Let k is a complete discrete valuation field with a uniformiser $\varpi_k \in k(1)$. We study relations between the semisimplicity of Banach k-algebras and that of their reductions, and between the semisimplicity of Banach modules and that of their reductions regarded as modules over the reduction of the operator algebras associated to them. As an application, we obtain an algorithm for determining whether a given finite dimensional unitary Banach k-linear representation is semisimple of finite type or not.

3.1 Reductions of Banach Algebras

Let $A \in \mathrm{ob}(\mathrm{Alg}^{\mathrm{Ban}}_{\mathrm{unit}}(k))$. We give a criterion of the simplicity of A by using the reduction.

Theorem 3.1. *If $\overline{A}$ is a finite dimensional simple $\overline{k}$-algebra (resp. a stably simple $\overline{k}$-algebra), then A is a finite dimensional simple unitary Banach k-algebra (resp. an orthogonally stably simple Banach k-algebra).*

Proof. Suppose that $\overline{A}$ is a finite dimensional simple $\overline{k}$-algebra. By Wedderburn's theorem (cf. [AF92] 13.4 Theorem), there is an isomorphism $\mathrm{M}_n(F) \to \overline{A}$ with $n \in \mathbb{N}$ in $\mathrm{Alg}(\overline{k})$ for some division $\overline{k}$-algebra F, and every simple left $\overline{A}$-module is isomorphic to $\lambda := \overline{A} \otimes_{\mathrm{M}_n(F)} F^n$. Since $\overline{A}$ is semisimple, every left $\overline{A}$-module is isomorphic to a direct sum of copies of λ.

We verify the simplicity of A. Since $\overline{A}$ is simple, it is non-zero, and hence so is A, Therefore A admits a maximal left ideal $\wp \subset A$. By Proposition 2.6, $A/\wp$ forms a simple Banach left A-module. By Proposition 1.12 (ii), the canonical projection $A \twoheadrightarrow A/\wp$ induces an isomorphism $\overline{A}/\overline{\wp} \to \overline{A/\wp}$ in $\mathrm{Vect}(\overline{A})$. Let $f \in \ker(\Pi_{A/\wp,A})$. Assume $f \neq 0$. By Proposition 1.13 (ii), there is a $c \in k^\times$ with $cf \in A(1) \setminus A(1-)$. Since $cf \in A(1)$ acts trivially on $(A/\wp)(1)$, so does $cf + A(1-) \in \overline{A}$ on $\overline{A/\wp}$. On the other hand, $\overline{A/\wp}$ is isomorphic to a direct sum of copies of λ in $\mathrm{Vect}(\overline{A})$, and hence $cf + A(1-)$ acts trivially on λ. It implies that $cf + A(1-)$ lies in the Jacobson radical of $\overline{A}$, which is trivial by the semisimplicity of $\overline{A}$. It contradicts $cf \notin A(1-)$. We obtain $f = 0$. Therefore $\Pi_{A/\wp,A}$ is injective, and the underlying left A-module of $A/\wp$ is a faithful simple left A-module. By Proposition 1.4, A is a finite dimensional simple unitary Banach k-algebra.

In addition, suppose $F = \overline{k}$. By Wedderburn's theorem, there is an isomorphism $\mathrm{M}_m(K) \to A$ with $m \in \mathbb{N} \setminus \{0\}$ in $\mathrm{Alg}(k)$ for some division k-algebra K. Since $\overline{A} \cong \mathrm{M}_n(\overline{k})$ admits an $(e_i)_{i=1}^n \in \mathrm{Idem}(\overline{A})^n$ satisfying $e_i \neq 0$ for any $i \in \mathbb{N} \cap [1,n]$ and $e_i e_j = 0$ for any $(i,j) \in (\mathbb{N} \cap [1,n])^2$ with $i \neq j$, $A(1)$ admits an $(E_i)_{i=1}^n \in \mathrm{Idem}(A(1))^n$ satisfying $E_i + A(1-) = e_i \neq 0$ for any $i \in \mathbb{N} \cap [1,n]$ and $E_i E_j = 0$ for any $(i,j) \in (\mathbb{N} \cap [1,n])^2$ with $i \neq j$ by [Azu51] Theorem 24. It implies $m \geq n$. We have $m^2 \dim_k D = \dim_k \mathrm{M}_m(D) = \dim_k A = \dim_{\overline{k}} \overline{A} = \dim_{\overline{k}} \mathrm{M}_n(\overline{k}) = n^2$ by Proposition 1.13 (ii) and (v). Therefore we obtain $m = n$ and $\dim_k K = 1$. It ensures that A is isomorphic in $\mathrm{Alg}(k)$ to $\mathrm{M}_n(k)$. We have $(1 - \sum_{i=1}^n E_i) + A(1-) = 0 \in \overline{A}$, and hence $1 - \sum_{i=1}^n E_i \in A(1-)$. By the orthogonality of $(E_i)_{i=1}^n$, $1 - \sum_{i=1}^n E_i$ is an idempotent. Since every element of $A(1-)$ is topologically nilpotent, we obtain $1 - \sum_{i=1}^n E_i = 0$. It implies $A(1) = \bigoplus_{i=1}^n E_i A(1) = \bigoplus_{i,j=1}^n E_i A(1) E_j$. Since $e_i \overline{A} e_j$ is isomorphic to $\overline{k}$ in $\mathrm{Vect}(\overline{k})$, $E_i A E_j$ is isomorphic to k in $\mathrm{Vect}^{\mathrm{Ban}}_{\leq 1}(k)$ by Proposition 1.12 (iii) and Proposition 1.13 (ii) and (v), and hence $E_i A(1) E_j$ is isomorphic to $k(1)$ in $\mathrm{Vect}(k(1))$ for any $(i,j) \in (\mathbb{N} \cap [1,n])^2$. Therefore $A(1)$ is isomorphic to $\mathrm{M}_n(k(1))$ in $\mathrm{Alg}(k(1))$ by [Azu51] Theorem 25. It implies that A is isomorphic to $\mathrm{M}_n(k)$ in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$. □

We verify a relation between the discretely spectral property and the reduction.

Theorem 3.2. *If $\overline{A}$ is a discretely spectral (resp. stably discretely spectral) $\overline{k}$-algebra, then A is a discretely spectral (resp. orthogonally stably discretely spectral) unitary Banach k-algebra. Moreover, the decomposition of A into simple (resp. orthogonally stably simple) Banach k-algebras is given as the localisation of the decomposition of $A(1)$ into indecomposable projective two-sided ideals.*

In order to verify Theorem 3.2, we introduce lifting properties of idempotents. It is well-known that for any finite group G, the maps $\mathrm{Idem}(k(1)[G]) \to \mathrm{Idem}(\overline{k}[G])$ and $\mathrm{Idem}(\mathrm{Z}(k(1)[G])) \to \mathrm{Idem}(\mathrm{Z}(\overline{k}[G]))$ induced by the natural projection $k(1)[G] \twoheadrightarrow \overline{k}[G]$ are surjective. We consider a generalisation of the fact.

Proposition 3.3. *Let B be a k-algebra, and $B_0 \subset B$ a $k(1)$-subalgebra with $\bigcap_{r=0}^{\infty} \varpi_k^r B_0 = \{0\}$. Then the map $\mathrm{Idem}(B_0) \to \mathrm{Idem}(B_0/\varpi_k B_0)$ induced by the canonical projection $B_0 \twoheadrightarrow B_0/\varpi_k B_0$ is surjective, and its restriction $\mathrm{Idem}(\mathrm{Z}(B_0)) \to \mathrm{Idem}(\mathrm{Z}(B_0/\varpi_k B_0))$ is bijective.*

Proof. The first assertion follows from [Azu51] Theorem 24. We verify the surjectivity of the map $\mathrm{Idem}(\mathrm{Z}(B_0)) \to \mathrm{Idem}(\mathrm{Z}(B_0/\varpi_k B_0))$. Let $\overline{e} \in \mathrm{Idem}(\mathrm{Z}(B_0/\varpi_k B_0))$. By the first assertion, there is an $E \in \mathrm{Idem}(B_0)$ with $E + \varpi_k B_0 = \overline{e}$. We show $E \in \mathrm{Idem}(\mathrm{Z}(B_0))$. Let $f \in B_0$. Assume $EfE \neq fE$. By $\bigcap_{r=0}^{\infty} \varpi_k^r B_0 = \{0\}$, there is a unique $r \in \mathbb{N}$ such that $EfE - fE \in \varpi_k^r B_0 \setminus \varpi_k^{r+1} B_0$. Put $f' := \varpi_k^{-r}(EfE - fE) \in B_0 \setminus \varpi_k B_0$. By $\overline{e} \in \mathrm{Z}(B_0/\varpi_k B_0)$, we have $Ef' - f'E \in \varpi_k B_0$. On the other hand, we have $Ef' = \varpi_k^{-r}(EfE - EfE) = 0$, $f'E = \varpi_k^{-r}(EfE - fE) = f'$, and hence $Ef' - f'E = -f'$. This contradicts $f' \notin \varpi_k B_0$. Therefore we obtain $EfE = fE$. Similarly the equality $EfE = Ef$ holds. We obtain $fE = EfE = Ef$. It implies $E \in \mathrm{Idem}(\mathrm{Z}(B_0))$. We show the injectivity. Let $(E, E') \in \mathrm{Idem}(\mathrm{Z}(B_0))^2$ with $E + \varpi_k B_0 = E' + \varpi_k B_0$. We have $E(1 - E') \in \mathrm{Idem}(\mathrm{Z}(B_0))$ by $EE' = E'E$ and $E(1 - E') = E - EE' = E(E - E') \in \varpi_k B_0$ by $E - E' \in \varpi_k B_0$. It ensures $E(1 - E') = 0$ by $\bigcap_{r=0}^{\infty} \varpi_k^r B_0 = \{0\}$. Similarly, we obtain $(1 - E)E' = 0$. We conclude $E = EE' = E'$. □

As a consequence, we obtain a lifting property of (central) idempotents for an unramified Banach k-algebra.

Corollary 3.4. *The reduction $A(1) \twoheadrightarrow \overline{A}$ induces a surjective map $\mathrm{Idem}(A(1)) \to \mathrm{Idem}(\overline{A})$ and bijective maps $\mathrm{Idem}(\mathrm{Z}(A(1))) \to \mathrm{Idem}(\mathrm{Z}(\overline{A}))$ and $\pi_0(A(1), k(1)) \to \pi_0(\overline{A}, \overline{k})$.*

Proof. Since A is unramified, we have $A(1-) = \varpi_k A(1)$ by Proposition 1.13 (ii). The assertion on $\mathrm{Idem}(-)$ follows from [Azu51] Theorem 24, and the assertion on $\mathrm{Idem}(\mathrm{Z}(-))$ follows from Proposition 3.3. Since the bijective map $\mathrm{Idem}(\mathrm{Z}(A(1))) \to \mathrm{Idem}(\mathrm{Z}(\overline{A}))$ preserves the order given by inclusions of the principal two-sided ideals generated by idempotents, it preserves the primitivity. Therefore it induces a bijective map $\pi_0(A(1), k(1)) \to \pi_0(\overline{A}, \overline{k})$ by Proposition 1.13. □

Proof of Theorem 3.2. Suppose that $\overline{A}$ is a discretely spectral (resp. stably discretely spectral) $\overline{k}$-algebra. It suffices to verify that $\hat{\Gamma}_{A,k} \colon A \to \hat{\Gamma}(A,k)$ is an isomorphism in $\mathrm{Alg}^{\mathrm{Ban}}_{\leq 1}(k)$, and $A/(1-E)A$ is simple (resp. stably simple) for any $E \in \pi_0(A(1),k(1))$. Let $E \in \pi_0(A(1),k(1))$. We have $A(1) = EA(1) \oplus (1-E)A(1)$, and hence $(1-E)A \cap A(1) = (1-E)A(1)$. Therefore the canonical projection $A \twoheadrightarrow A/(1-E)A$ induces an isomorphism $\iota_E \colon A(1)/(1-E)A(1) \to (A/(1-E)A)(1)$, and $\overline{A/(1-E)A}$ is isomorphic in $\mathrm{Alg}(\overline{k})$ to $\overline{A}/(1-e)\overline{A}$. In particular, $\overline{A/(1-E)A}$ is isomorphic in $\mathrm{Alg}(\overline{k})$ to $\overline{A}/(1-e)\overline{A}$ by Proposition 1.12 (iii), and hence $A/(1-E)A$ is simple (resp. orthogonally stably simple) by Theorem 3.1. We have $\hat{\Gamma}(A,k)(1) = \Gamma(A(1),k(1))$ by definition. Therefore the inclusion $\Gamma(A(1),k(1)) \hookrightarrow \hat{\Gamma}(A,k)$ induces an isomorphism $\iota \colon k \otimes_{k(1)} \Gamma(A(1),k(1)) \to \hat{\Gamma}(A,k)$ in $\mathrm{Alg}(k)$. The composite $\hat{\Gamma}_{A,k} \circ \iota$ coincides with the localisation of $\Gamma_{A(1),k(1)} \colon A(1) \to \Gamma(A(1),k(1))$. Therefore it suffices to verify the bijectivity of $\Gamma_{A(1),k(1)}$, because the norm on an unramified Banach k-vector space is uniquely determined by its closed unit disc.

Let $f \in \ker(\Gamma_{A(1),k(1)})$. Assume $f \neq 0$. By Proposition 1.13 (ii), there is a $c \in k(1) \setminus \{0\}$ with $|c| = \|f\|$. Then we have $c^{-1}f \in A(1) \setminus A(1-)$ and hence $c^{-1}f + A(1-) \neq 0 \in \overline{A}$. Since $\overline{A}$ is discretely spectral, there is an $e \in \pi_0(\overline{A},\overline{k})$ with $c^{-1}f + A(1-) \notin (1-e)\overline{A}$. By Corollary 3.4, there is an $E \in \pi_0(A(1),k(1))$ with $E + A(1-) = e$. By $c^{-1}f \in c^{-1}\ker(\Gamma_{A(1),k(1)}) \cap A(1) \subset \ker(\hat{\Gamma}_{A,k}) \cap A(1) = \ker(\Gamma_{A(1),k(1)})$, we have $c^{-1}f \in (1-E)A(1)$. It contradicts $c^{-1}f + A(1-) \notin (1-e)\overline{A}$. It implies $f = 0$. Therefore $\Gamma_{A(1),k(1)}$ is injective.

Let $\varphi_0 = (\varphi_E)_{E \in \pi_0(A(1),k(1))} \in \Gamma(A(1),k(1))$. By Corollary 3.4, the reduction $A(1) \twoheadrightarrow \overline{A}$ induces a bijective map $\iota \colon \pi_0(A(1),k(1)) \to \pi_0(\overline{A},\overline{k})$. Since $\overline{A}$ is discretely spectral, there is an $f_1 \in A(1)$ with $\Gamma_{\overline{A},\overline{k}}(f_1 + A(1-)) = ((\varphi_{\iota^{-1}(e)} + A(1-)) + (1-e)\overline{A})_{e \in \pi_0(\overline{A},\overline{k})} \in \Gamma(\overline{A},\overline{k})$. Then we have $\varphi_0 - \Gamma_{A(1),k(1)}(f_1) = c_1\varphi_1$ for a $(c_1,\varphi_1) \in k(1-) \times \Gamma(A(1),k(1))$. Replacing φ_0 by φ_1, we obtain an $f_2 \in A(1)$ with $\varphi_1 - \Gamma_{A(1),k(1)}(f_2) = c_2\varphi_2$ for a $(c_2,\varphi_2) \in k(1-) \times \Gamma(A(1),k(1))$. Repeating this process, we obtain a sequence $(f_i,c_i,\varphi_i)_{i=1}^{\infty} \in (A(1) \times k(1-) \times \Gamma(A(1),k(1)))^{\mathbb{N}\setminus\{0\}}$ with $\varphi_i - \Gamma_{A(1),k(1)}(f_{i+1}) = c_{i+1}\varphi_{i+1}$ for any $i \in \mathbb{N}$. Since the valuation of k is discrete, we have $\lim_{i\to\infty} |\prod_{j=1}^{i-1} c_j| = 0$. Therefore $\sum_{i=1}^{\infty}(\prod_{j=1}^{i-1} c_j)f_i$ converges to an $f \in A(1)$ by the completeness of A and the strong triangle inequality, and $\Gamma_{A(1),k(1)}(f)$ coincides with $\sum_{i=1}^{\infty}(\prod_{j=1}^{i-1} c_j)\Gamma_{A(1),k(1)}(f_i) = \sum_{i=0}^{\infty}(\prod_{j=1}^{i} c_j)(\varphi_i - c_{i+1}\varphi_{i+1}) = \varphi_0$ by the continuity of $\Gamma_{A(1),k(1)}$. Thus $\Gamma_{A(1),k(1)}$ is bijective. □

By Theorem 3.2 and Corollary 3.4, we obtain the following:

Corollary 3.5. *If $\overline{A}$ is a semisimple (resp. stably semisimple) $\overline{k}$-algebra, then A is a semisimple (resp. orthogonally stably semisimple) unitary Banach k-algebra. Moreover, the decomposition of A into simple (resp. orthogonally stably simple) Banach k-algebras is given as the localisation of the decomposition of $A(1)$ into indecomposable projective two-sided ideals.*

3.2 Reductions of Banach Modules

Let $V \in \mathrm{ob}(\mathrm{Vect}^{\mathrm{Ban}}_{\mathrm{unit}}(A))$. The following is obvious but is remarkable because the same does not necessarily hold when we consider a submetric Banach left A-module which is not unitary:

Proposition 3.6. *If $\overline{V}$ is a simple left $\overline{\mathrm{C}^*(A,V)}$-module, then V is a simple unitary Banach left A-module.*

Proof. Since $\overline{V}$ is a simple left $\overline{\mathrm{C}^*(A,V)}$, we have $\overline{V} \neq \{0\}$. Therefore we obtain $V \neq \{0\}$. Let $L \subset V$ be a closed left A-submodule with $L \neq \{0\}$. By Proposition 1.11 (i) and Proposition 1.13 (ii), we have $\overline{L} \neq \{0\}$. Therefore the embedding $\overline{L} \hookrightarrow \overline{V}$ induced by the inclusion $L \hookrightarrow V$ is an isomorphism in $\mathrm{Vect}(\overline{\mathrm{C}^*(A,V)})$ by the assumption. Assume $L \neq V$. Let $v \in V \setminus L$. Since L is closed, there is an $r \in \mathbb{N} \setminus \{0\}$ such that $\|v - w\| > |\varpi_k|^r \|v\|$ for any $w \in L$. By Proposition 1.13 (ii), there is a $c_1 \in k^\times$ with $\|v\| = |c_1|$. Take a $v_1 \in L$ with $\|v_1\| = 1$ and $c_1^{-1} v + V(1-) = v_1 + V(1-) \in \overline{V}$. By $v \in V \setminus L$, we have $v - c_1 v_1 \in V \setminus L$. Replacing v by $v - c_1 v_1$, we obtain a $(c_2, v_2) \in k^\times \times L$ with $\|v - c_1 v_1\| = |c_2|$ and $c_2^{-1}(v - c_1 v_1) + V(1-) = v_2 + V(1-) \in \overline{V}$. Repeating this process, we obtain a sequence $(c_i, v_i)_{i=1}^r \in (k^\times \times L)^r$ with $\|v - \sum_{i=1}^{j-1} c_i v_i\| = |c_j|$ and $c_j^{-1}(v - \sum_{i=1}^{j-1} c_i v_i) + V(1-) = v_j + V(1-) \in \overline{V}$ for any $j \in \mathbb{N} \cap [1, r]$. Since V is unitary, we have $V(1-) = \varpi_k V(1)$. It ensures $\|v - \sum_{i=1}^r c_i v_i\| \leq |\varpi_k|^r \|v\|$. This contradicts the choice of r. It implies $L = V$. Thus V is a simple unitary Banach left A-module. □

We have a criterion of the isotypic property of a unitary Banach left A-module by using the reduction.

Proposition 3.7. *If $\overline{V}$ is an isotypic (resp. a stably isotypic) left $\overline{\mathrm{C}^*(A,V)}$-module admitting a finite dimensional simple (resp. absolutely simple) left $\overline{\mathrm{C}^*(A,V)}$-submodule, then V is an isotypic (resp. a stably isotypic) unitary Banach left A-module admitting a finite dimensional simple (resp. absolutely simple) Banach left A-submodule.*

Proof. Put $B := \mathrm{C}^*(A,V)$. Suppose that $\overline{V}$ is an isotypic (resp. resp. a stably isotypic) left $\overline{B}$-module admitting a finite dimensional simple left $\overline{B}$-submodule. Since B is a closed k-subalgebra of $\mathscr{B}(V)$, we have $\mathrm{Ann}_{\overline{B}}(\overline{V}) = \{0\}$ by Proposition 2.1. Therefore $\overline{B}$ is a finite dimensional simple (resp. stably simple) $\overline{k}$-algebra by Proposition 1.4. By Theorem 3.1, B is a finite dimensional simple (resp. an orthogonally stably simple) Banach k-algebra with $dim_k B = \dim_{\overline{k}} \overline{B}$. Therefore V is an isotypic (resp. a stably isotypic) unitary Banach left A-module admitting a finite dimensional simple (resp. absolutely simple) Banach left A-submodule by Proposition 2.9. □

As a consequence, we obtain a criterion of the finite orthogonal type property of V.

Corollary 3.8. *If $\overline{V}$ is a semisimple (resp. stably semisimple) left $\overline{\mathrm{C}^*(A,V)}$-module of finite type and every simple (resp. absolutely simple) left $\overline{\mathrm{C}^*(A,V)}$-submodule of $\overline{V}$ is finite dimensional, then V is a semisimple (resp. stably semisimple) unitary Banach left A-module of finite orthogonal type such that every simple (resp. absolutely simple) Banach left A-submodule of V is finite dimensional.*

Proof. If $V = \{0\}$, then V is stably semisimple of finite orthogonal type admitting no simple Banach left A-submodule. Assume $V \neq \{0\}$. Put $B := \mathrm{C}^*(A, V)$. Suppose that $\overline{V}$ is a semisimple (resp. stably semisimple) left $\overline{B}$-module of finite type and every simple (resp. absolutely simple) left $\overline{B}$-submodule of $\overline{V}$ is finite dimensional. By Proposition 2.1, B is a unitary Banach k-algebra with $\mathrm{Ann}_{\overline{B}}(\overline{V}) = \{0\}$. Therefore $\overline{B}$ is a finite dimensional semisimple $\overline{k}$-algebra (resp. stably semisimple $\overline{k}$-algebra) by Corollary 1.8. It ensures that B is a finite dimensional semisimple (resp. stably semisimple) unitary Banach k-algebra and the decomposition of B into simple (resp. orthogonally stably simple) Banach k-algebras is given as the localisation of the decomposition of $B(1)$ into indecomposable projective two-sided ideals by Corollary 3.5. Therefore B coincides with $\Pi_{V,A}(A)$ by Proposition 1.12 (i), and V is isomorphic in $\mathrm{Ban}^{\mathrm{unit}}(B)$ to the direct sum of a finite family of isotypic (resp. stably isotypic) unitary Banach left B-modules by Proposition 2.5 and Proposition 3.7. It implies that V is semisimple (resp. stably semisimple) of finite orthogonal type. □

We recall that the aim of this paper is to establish an algorithm for determining a variant of the semisimplicity of a unitary Banach k-linear representation of a topological monoid. Although the converses of the propositions above do not necessarily hold, we have a criterion for the orthogonal stably semisimplicity.

Proposition 3.9. *Suppose that $\overline{V}$ is finite dimensional. Then V is an absolutely simple (resp. orthogonally stably isotypic, orthogonally stably semisimple) unitary Banach left A-module if and only if $\overline{V}$ is an absolutely simple (resp. a stably isotypic, a stably semisimple) left $\overline{\mathrm{C}^*(A, V)}$-module.*

Proof. If that V is absolutely simple, then $\overline{V}$ is isomorphic in $\mathrm{Vect}(\overline{\mathrm{C}^*(A, V)}) \otimes_{\mathrm{M}_n(\overline{k})} \overline{k}^n$ for an isomorphism $\mathrm{M}_n(k) \to \mathrm{C}^*(A, V)$ in $\mathrm{Alg}^{\mathrm{Ban}}_{\mathrm{unit}}(k)$ by Lemma 2.10, and hence is an absolutely simple left $\overline{Cst(A, V)}$-module by Proposition 1.6. Therefore if V is an orthogonally stably isotypic (resp. orthogonally stably semisimple) unitary Banach left A-module, then $\overline{V}$ is a stably isotypic (resp. stably semisimple) left $\overline{\mathrm{C}^*(A, V)}$-module because $\overline{V}$ is isomorphic in $\mathrm{Vect}(\overline{\mathrm{C}^*(A, V)})$ with the direct sum in $\mathrm{Vect}(\overline{\mathrm{C}^*(A, V)})$ of the reductions of the absolutely simple unitary Banach A-submodules appearing in an orthogonal direct sum decomposition of V.

If $\overline{V}$ is a stably isotypic (resp. stably semisimple) left $\overline{\mathrm{C}^*(A, V)}$-module, V is an orthogonally stably isotypic (resp. orthogonally stably semisimple) unitary Banach left A-module by Proposition 3.7. Suppose that $\overline{V}$ is an absolutely simple left $\overline{\mathrm{C}^*(A, V)}$-module. Then V is an orthogonally stably isotypic unitary Banach left A-module. Since $\overline{V}$ admits exactly two left $\mathrm{C}^*(A, V)$-submodules, V admits exactly two closed left $\mathrm{C}^*(A, V)$-submodules by Proposition 1.12 (ii) and Proposition 1.13 (ii) and (v). Therefore V is an absolutely simple unitary Banach left A-module. □

Finally, we consider a criterion of the semisimplicity of V by using the reduction of the bigger operator algebra $\mathrm{W}^*(A, V)$.

Proposition 3.10. *If $\overline{\mathrm{W}^*(A,V)}$ is a discretely spectral (resp. stably discretely spectral) $\overline{k}$-algebra and $\overline{V}$ vanishes at infinity as a left $\overline{\mathrm{W}^*(A,V)}$-module, then V is a semisimple (resp. stably semisimple) unitary Banach left A-module.*

Proof. Put $B := \mathrm{W}^*(A,V)$. Suppose that $\overline{B}$ is a discretely spectral (resp. stably discretely spectral) $\overline{k}$-algebra. By Theorem 3.2, B is a discretely spectral (resp. orthogonally stably discretely spectral) unitary Banach k-algebra and the decomposition of B into simple (resp. orthogonally stably simple) Banach k-algebras is given as the localisation of the decomposition of $B(1)$ into indecomposable projective two-sided ideals. We show that V vanishes at infinity as a unitary Banach left B-module. Let $v \in V\backslash\{0\}$. By Proposition 1.13 (ii), there is a $c \in k^\times$ with $\|v\| = |c|$. By the injectivity of $\eta_{\overline{V},\overline{B},\overline{k}}$, there is an $e \in \pi_0(\overline{B},\overline{k})$ with $e(c^{-1}v + V(1-)) \neq 0 \in \overline{V}$. By Corollary 3.4, there is an $E \in \pi_0(B(1),k(1))$ with $E + A(1-) = e$. We obtain $Ev = c(Ec^{-1}v) \neq 0$. It implies that $\hat{\eta}_{V,B,k}$ is injective. Therefore V is semisimple (resp. stably semisimple) by Proposition 2.14. □

3.3 Reductions of Banach Representations

We apply the results in §3.2 to a unitary Banach k-linear representation of a topological monoid G. Since the irreducibility and the semisimplicity of a given unitary Banach k-linear representation of G are preserved even if the topology on G is replaced by the discrete topology, we start with a unitary Banach k-linear representation of a discrete monoid.

Let G_0 be a monoid. We endow G_0 with the discrete topology so that the notion of a unitary Banach k-linear representation of G_0 makes sense. The k-algebra structure of $k[G_0]$ extends to $\mathrm{C}_0(G_0,k)$ through the natural embedding $k[G_0] = k^{\oplus G_0} \hookrightarrow \mathrm{C}_0(G_0,k)$, and $\mathrm{C}_0(G_0,k)$ forms a Banach k-algebra. The induced multiplication $\mathrm{C}_0(G_0,k) \times \mathrm{C}_0(G_0,k) \to \mathrm{C}_0(G_0,k)$, $(f,f') \mapsto f * f'$ can be described in the following way: Let $((f,f'),g) \in \mathrm{C}_0(G_0,k)^2 \times G_0$. Put $H_g := \{(h,h') \in G_0^2 \mid hh' = g\}$. Then $(f * f')(g)$ is given as the limit $\sum_{(h,h')\in H_g} f(h)f'(h')$ of the converging net $(\sum_{(h,h')\in S} f(h)f'(h'))_{S\subset H_g, \#S<\infty}$ indexed by the set of finite subsets $S \subset H_g$ ordered by inclusions. We note that $\mathrm{C}_0(G_0,k)$ coincides with the set of functions $f \colon G_0 \to k$ with $\#\{h \in G_0 \mid |f(h)| > \epsilon\} < \infty$ for any $\epsilon \in (0,\infty)$, and hence $\{h \in G_0 \mid f(h) \neq 0\}$ is a countable set for any $f \in \mathrm{C}_0(G_0,k)$. Therefore $\sum_{(h,h')\in H_g} f(h)f'(h')$ is an essentially countable sum for any $((f,f'),g) \in \mathrm{C}_0(G_0,k)^2 \times G_0$. By the universality of the direct sum in $\mathrm{Vect}_{\leq 1}^{\mathrm{Ban}}(k)$ and the continuity of the natural embedding $G_0 \hookrightarrow \mathrm{C}_0(G_0,k)$, we obtain a natural equivalence $\hat{\phi}_{G_0,k} \colon \mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(G_0,k) \to \mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(\mathrm{C}_0(G_0,k))$ preserving the underlying Banach k-vector space.

Let $(V,\rho) \in \mathrm{ob}(\mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(G_0,k))$. We put $\mathrm{C}^*(G,(V,\rho)) := \mathrm{C}^*(\mathrm{C}_0(G_0,k), \hat{\phi}_{G_0,k}(V,\rho))$. Since the underlying Banach k-vector space of $\hat{\phi}_{G_0,k}(V,\rho)$ is V, $\mathrm{C}^*(G,(V,\rho))$ is unitary by Proposition 2.1, and (V,ρ) is irreducible (resp. isotypic, semisimple of finite orthogonal type, semisimple, absolutely irreducible, stably isotypic, stably semisimple of finite orthogonal type, stably semisimple, orthogonally stably isotypic, orthogonally stably semisimple) if and only is $\hat{\phi}_{G_0,k}(V,\rho)$ is simple (resp. isotypic, semisimple of finite orthogonal type, semisimple, absolutely simple, stably isotypic, stably semisimple of finite orthogonal type, stably semisimple, orthogonally stably isotypic, orthogonally stably semisimple). Therefore by Remark 1.10, Proposition 3.7, Proposition 3.9, Corollary 3.5, and Corollary 1.5, we obtain the following:

Proposition 3.11. *The following hold:*

(i) If $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$ is a finite dimensional simple left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-module, then (V,ρ) is a finite dimensional irreducible unitary Banach k-linear representation of G_0.

(ii) If $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$ is an isotypic left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-module admitting a finite dimensional simple left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-submodule, then (V,ρ) is an isotypic (resp. a stably isotypic) unitary Banach k-linear representation of G_0.

(iii) If $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$ is a semisimple left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-module of finite type and every simple left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-submodule of $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$ is finite dimensional, then (V,ρ) is a semisimple unitary Banach k-linear representation of G_0 of finite orthogonal type.

(iv) Suppose that $\overline{V}$ is finite dimensional. Then (V,ρ) is an absolutely irreducible (resp. orthogonally stably isotypic, orthogonally stably semisimple) unitary Banach k-linear representation of G_0 if and only if $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$ is an absolutely simple (resp. a stably isotypic, a stably semisimple) left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-module.

The map $\overline{k}[G_0] \to \overline{\mathrm{C}^*(G_0,(V,\rho))}$ induced by the isomorphism $\overline{k}[G_0] \to \overline{\mathrm{C}_0(G_0,k)}$ in Proposition 1.11 (i) is not necessarily surjective, and hence the left $\overline{\mathrm{C}^*(G_0,(V,\rho))}$-module structure on $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$ possibly possesses more information than the structure of the reduction $\overline{(V,\rho)}$ as the smooth $\overline{k}$-linear representation of G_0. One of the critical problem in applications of Proposition 3.11 is the difficulty of the computation of $\overline{\hat{\phi}_{G_0,k}(V,\rho)}$. In order to avoid the difficulty, we restrict the classes of k and topological monoids.

Definition 3.12. A profinite group G is said to be a *p-adic group* if G admits a decreasing sequence $(G_r)_{r\in\mathbb{N}}$ of open normal pro-p subgroups of G such that G_r is contained in the closure of the normal subgroup generated by $\{g^{p^r} \mid g \in G_0\}$ for any $r \in \mathbb{N}$.

For example, for a profinite group G, if there is a pair (H,x) of an open pro-p subgroup $H \subset G$ and a sequence $x = (x_i)_{i=1}^d \in H^d$ with $d \in \mathbb{N}$ such that every $g \in H$ is presented as the ordered product $\prod_{i=1}^d x_i^{a_i}$ with $(a_i)_{i=1}^d \in \mathbb{Z}_p^d$, then G is p-adic. In particular, we obtain the following by [Laz65] II 2.2.3 Lemme, II 2.2.6 Proposition, and III 3.1.3 Proposition:

Example 3.13. Every compact p-adic Lie group is a p-adic group.

Henceforth, suppose that k is a local field, i.e. a complete discrete valuation field with $\#\overline{k} < \infty$. We denote by $\tau_k \colon \overline{k} \hookrightarrow k(1)$ the unique Teichmüller lift. Let G be a p-adic group. We fix a decreasing sequence $(G_r)_{r\in\mathbb{N}}$ of open normal pro-p subgroups of G such that G_r is contained in the closure of the normal subgroup generated by $\{g^{p^r} \mid g \in G_0\}$ for any $r \in \mathbb{N}$. Take a complete system $\Gamma_0 \subset G$ of representatives of the canonical projection $G \twoheadrightarrow G/G_0$ with $1 \in \Gamma_0$. For an $s \in \mathbb{N}$ and a $\Gamma \subset G$, we denote by $A_{\Gamma,s} \subset k(1)[G]$ the image of the map $(\overline{k}^{s+1})^{\oplus \Gamma} \to k(1)[G]$, $((c_{g,i})_{i=0}^{s})_{g \in \Gamma} \mapsto \sum_{g \in \Gamma} (\sum_{i=0}^{s} \tau_k(c_{g,i}) \varpi_k^i)[g]$, which is a finite set as long as so is Γ.

In order to describe algorithms, we prepare the convention. We denote by Rep the class of pairs $(n,(V,\rho))$ of an $n \in \mathbb{N} \setminus \{0\}$ and a $(V,\rho) \in \mathrm{ob}(\mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(G,k))$ with $\dim_k V = n$. Let $n \in \mathbb{N}$. For an $s \in \mathbb{N}$, we denote by $r_{n,s} \in \mathbb{N}$ the integer such that $p^{r_{n,s}}$ is the maximum of the orders of p-torsion elements of $\mathrm{GL}_n(k(1)/\varpi_k^{s+1}k(1))$. Let $(k^n,\rho) \in \mathrm{ob}(\mathrm{Vect}_{\mathrm{unit}}^{\mathrm{Ban}}(G,k))$. We put $I_\rho := \{f \in k[G] \mid \mathrm{BS}(\rho)(f)(V(1)) \subset V(1)\}$. By the definition of $r_{n,s}$, $\{g-1 \mid g \in G_{r_{n,s}}\} \subset k(1)[G]$ is contained in $A_{G,0} \cap \varpi_k^s I_\rho$. We denote by $\overline{\mathrm{BS}}(\rho) \colon I_\rho/\varpi_k I_\rho \to \mathrm{M}_n(\overline{k})$ the natural extension of $\mathrm{BS}(\overline{\rho})$. For a finite subset $S \subset I_\rho$, we denote by $B_S \subset \mathrm{M}_n(\overline{k}^n)$ the $\overline{k}$-algebra generated by $\{\overline{\mathrm{BS}}(\rho)(f) \mid f \in S\}$. We introduce two processes (R1) and (R2).

The input data for the process (R1) is an $(n,(V,\rho)) \in \mathrm{Rep}$ with $V = k^n$. The process (R1) runs in the following way:

(i) Put $R := \{1\} \subset G$, $\Gamma := \Gamma_0 \subset G$, $s := 0 \in \mathbb{N}$, and $S := A_{\Gamma_0,0} \subset I_\rho$.

(ii) If $B_S = \mathrm{M}_n(\overline{k})$, then go to (v).

(iii) Replace R by a complete system of representatives of the canonical projection $G_s \twoheadrightarrow G_s/G_{s+1}$, Γ by the finite set $\{gh \mid (g,h) \in \Gamma \times R\}$, s by $s+1$, and S by the finite set $\{f \in \varpi_k^{-s} A_{\Gamma,s} \mid \exists f' \in S \cap \ker(\overline{BS}(\rho)), \varpi_k f - f' \in \varpi_k k(1)[G] + \sum_{h \in R} \varpi_k^{-s+1} k(1)[G](h-1)\}$.

(iv) Go to (ii).

(v) Stop the process.

In the process (R1), S grows as finite subsets of I_ρ, and B_S grows as $\overline{k}$-subalgebras of $\mathrm{M}_n(\overline{k})$. Since $\mathrm{M}_n(\overline{k})$ is finite dimensional, B_S forms an increasing sequence which is eventually constant. By Proposition 1.6 and Proposition 3.11 (iv), we obtain the following:

Theorem 3.14. *The process (R1) stops if and only if (V,ρ) is an absolutely simple unitary Banach k-linear representation of G.*

The input data for the process (R2) is an $(n,(V,\rho)) \in \mathrm{Rep}$. The process (R2) runs in the following way:

(i) Replace $(n,(V,\rho))$ by $(n,(V',\rho')) \in \mathrm{Rep}$ with $V' = k^n$ such that (V,ρ) is isometrically isomorphic to (V',ρ').

(ii) Put $R := \{1\} \subset G$, $\Gamma := \Gamma_0 \subset G$, $s := 0 \in \mathbb{N}$, $S := A_{\Gamma_0,0} \subset I_\rho$, and $E := 1 \in \mathrm{Idem}(\mathrm{M}_n(k(1)))$.

(iii) If B_S is a stably simple $\overline{k}$-algebra, then go to (vii).

(iv) If B_S is not stably semisimple $\overline{k}$-algebra, then go to (ix).

(v) Search the finite set B_S for an $e \in \pi_0(B_S, \overline{k})$, take an element E_0 of the $k(1)$-subalgebra of $\mathrm{M}_n(k(1))$ generated by $\{\mathrm{BS}(\rho)(s) \mid s \in S\}$ with $E_0 + \mathrm{M}_n(k)(1-) = e$, and replace E by the limit $E' \in \mathrm{Idem}(\mathrm{M}_n(k(1)))$ of the sequence $(E_n)_{n \in \mathbb{N}} \in \mathrm{M}_n(k(1))$ defined by $E_{n+1} := 3E_n^2 - 2E_n^3$ for each $n \in \mathbb{N}$.

(vi) If E is G-equivariant, then go to (xi). Otherwise, go to (ix).

(vii) If $\mathrm{rank}_{B_S} \overline{V} = 1$, then go to (xii).

(viii) Search the finite set $\overline{V}^{\mathrm{rank}_{B_S} \overline{V}}$ for a B_S-linear basis $(\overline{v}_j)_{j=1}^{\mathrm{rank}_{B_S} \overline{V}}$ of the free left B_S-module $\overline{V}$, and take a lift $(v_j)_{j=1}^{\mathrm{rank}_{B_S} \overline{V}} \in V^{\mathrm{rank}_{B_S} \overline{V}}$. If $k(1)[G]v_1 \cap \sum_{j=2}^{\mathrm{rank}_{B_S} \overline{V}} k(1)[G]v_j = \{0\}$, then replace E by the projection $V = k[G]v_1 \oplus \sum_{j=2}^{\mathrm{rank}_{B_S} \overline{V}} k[G]v_j \twoheadrightarrow k[G]v_1$, and go to (xi).

(ix) Replace R by a complete system of representatives of the canonical projection $G_s \twoheadrightarrow G_s/G_{s+1}$, Γ by the finite set $\{gh \mid (g, h) \in \Gamma \times R\}$, s by $s + 1$, and S by the finite set $\{f \in \varpi_k^{-s} A_{\Gamma,s} \mid \exists f' \in S \cap \ker(\overline{BS}(\rho)), \varpi_k f - f' \in \varpi_k k(1)[G] + \sum_{h \in R} \varpi_k^{-s+1} k(1)[G](h-1)\}$.

(x) Go to (iii).

(xi) Output $(\dim_k EV, (EV, \rho))$ and $(\dim_k (1-E)V, ((1-E)V, \rho))$, and go to (xiii).

(xii) Output $(n, (V, \rho))$.

(xiii) Stop the process.

We note that the step (v) is a well-known construction of a lift of an idempotent (cf. the proof of [Mih16] Theorem 3.11). By Corollary 3.4, Corollary 3.5, and Proposition 3.11 (iv), we obtain the following:

Proposition 3.15. *The process (R2) stops if and only if (V, ρ) is an absolutely simple unitary Banach k-linear representation of G or is the orthogonal direct sum of two non-zero G-stable k-vector subspaces.*

We establish an algorithm (RR) for determining whether a unitary Banach k-linear representation of G is orthogonally stably semisimple or not. The input data for the process (RR) is an $(n, (V, \rho)) \in \mathrm{Rep}$. The process (RR) runs in the following way:

(i) Put $m := 1 \in \mathbb{N} \backslash \{0\}$, $i := 1 \in \mathbb{N} \cap [1, m]$, $t := 0 \in \mathbb{N}$, $\sigma = (\sigma_i)_{i=1}^m := (n, (V, \rho)) \in \mathrm{Rep}^m$, and $\Sigma := (\sigma_1, \sigma_1) \in \mathrm{Rep}^2$.

(ii) Execute (R2) with input data σ_i, replace t by the number of the outputs, and $\Sigma \in \mathrm{Rep}^2$ by the ordered pair of the outputs if $t = 2$.

(iii) If $t = 1$, then go to (vi).

(iv) Replace m by $m + 1$, and σ by the sequence $\sigma' = (\sigma'_j)_{j=1}^m \in \mathrm{Rep}^m$ given by setting $\sigma'_j := \sigma_j$ for each $j \in \mathbb{N} \cap [1, i - 1]$, $(\sigma'_i, \sigma'_{i+1}) := \Sigma$, and $\sigma'_j := \sigma_{j-1}$ for each $j \in \mathbb{N} \cap [i + 2, m]$.

(v) Go to (ii).

(vi) If $i = m$, then go to (ix).

(vii) Replace i by $i + 1$.

(viii) Go to (ii).

(ix) Output σ.

(x) Stop the process.

By Proposition 3.15, we obtain the following:

Theorem 3.16. *The algorithm (RR) stops if and only if (V, ρ) is orthogonally stably semisimple, in which case (V, ρ) is isometrically isomorphic to the orthogonal direct sum of the unitary Banach k-linear representations of G in the output.*

Theorem 3.16 is a generalisation of [Mih16] Theorem 3.23 on the algorithm with a repetition of reductions for determining whether a given matrix over k is diagonalisable by a unitary matrix (cf. [Mih16] p. 762) or not. Indeed, the unitary diagonalisability of an $M \in \mathrm{M}_n(k)$ is equivalent to the orthogonal stable semisimplicity of the unitary Banach k-linear representation $\mathbb{Z}_p \times k^n \to k^n$, $(a, v) \mapsto (1 + \varpi_k^{r+1} M)^a v$ of $\mathbb{Z}_p$ for a sufficiently large $r \in \mathbb{N}$ with $\varpi_k^r M \in \mathrm{M}_n(k(1))$.

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